Skewness-Adjusted Binomial Interest Rate Models

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In this paper, we illustrate how a skewness-adjusted binomial model can be used to calibrate a binomial interest rate tree for increasing and decreasing interest rate cases. We then show how the implied yield curves and implied forward rates generated from the model result in flat, normal, and inverted yield curves that are consistent with the end-of-the-period distribution. We conclude the paper by showing how skewness can be incorporated into the Black, Derman, and Toy (BDT) calibration model and showing the possible mispricing that can result when the BDT variability conditions are not adjusted to reflect skewness.

INTRODUCTION

The binomial interest rate tree has become an important model to the valuation of bonds, interest rate options, and other interest-sensitive securities. There are two general approaches to modeling stochastic interest rate movements using a binomial model. The first is to estimate the up (u) and down (d) parameters based on the mean and variability of the underlying interest rate (see Rendelman and Bartter (1980) and Cox, Ingersoll, and Ross (1985)). Because the bond values generated from this model usually differ from equilibrium prices (prices obtained by discounting cash flow by spot rates), additional assumptions regarding risk premiums must be made to explain market prices. The second approach to modeling interest rate movements is to calibrate the tree to the current spot yield curve (see Black, Derman, and Toy (1990), Ho and Lee (1986), and Heath, Jarrow, and Morton (1992)). This calibration method generates a binomial tree that is consistent with an estimated relationship between the variance of the upper and lower spot rates and yields a bond value that reflects the current term structure. Since the resulting binomial tree is synchronized with current spot rates, this model yields values for option-free bonds that are equal to their equilibrium prices. A calibration model has the property that if the assumption regarding the evolution of interest rates is correct, then the model's bond price and derivative values are supported by arbitrage arguments.

Both models assume that the interest rate's logarithmic return is normally distributed. In a 2006, study, Johnson, Zuber, and Gandar applied the D'Agostino, Belanger, and D'Agostino tests of normality to statistically test for the significance of skewness for the logarithmic returns for the 1-year, 2-year, 5-year and 10-year U.S. Treasury yields (D'Agostino, et. al., 1999). They, in turn, found a large number of periods of increasing and decreasing interest rate cases in which skewness was significant. Their findings, in turn, point to the importance of using a skewness-adjusted binomial interest rate model when pricing bond and bond derivatives when the underlying interest rate is expected to increase or decrease. In this paper, we show how the Johnson, Pawlukiewicz, and Mehta skewness-adjusted model can be used to calibrate a binomial interest rate tree for stable, increasing and decreasing interest rate cases in which the end-of-the period distribution is characterized by skewness (Johnson, et. al., 1997). We then show how the implied yield curves and implied forward rates generated from the model result in flat, normal, and inverted yield curves that are consistent with the end-of-the-period distribution. Using the skewness-adjusted binomial equilibrium model, we then compare the prices of option-free bonds, spot options, bond futures, and futures options under stable, increasing, and decreasing rate scenarios. Finally, we conclude the paper by showing how skewness can be incorporated into the Black, Derman, and Toy (BDT) calibration model and showing the possible mispricing that can result when the BDT variability conditions are not adjusted to reflect skewness.

SKEWNESS-ADJUSTED BINOMIAL OPTION PRICING MODEL

A binomial process that converges to an end-of-the-period distribution of logarithmic returns that is normal will have equal probabilities of the underlying security increasing (q) or decreasing (1 – q) each period, whereas one that converges to a distribution that is skewed will not. In the Johnson, Pawlukiewicz, and Mehta (JPM) model, the upward (u), downward parameter (d), and q values defining a binomial process are found by setting the equations for the binomial distribution's expected value, variance, and skewness equal to their respective empirical values, then solving the resulting equation system simultaneously for u, d, and q:

$$E(g_n) = nE(g_1) = n[q \ln u + (1-q)\ln d] = \mu_e \tag{1}$$

$$V(g_n) = nq(1-q)[\ln(u/d)]^2 = V_e \tag{2}$$

$$Sk(g_n) = n[q(1-q)^3 - q^3(1-q)][\ln(u/d)]^3 = \delta_e \tag{3}$$

where:

$$E(g_n) = \sum_{j=0}^{n} p_{nj}\, g_{nj} = nE(g_1) = n[q \ln u + (1-q)\ln d]$$

$$V(g_n) = E[g_n - E(g_n)]^2 = nV(g_1) = nq(1-q)[\ln(u/d)]^2$$

$$Sk(g_n) = E[g_n - E(g_n)]^3 = nSk(g_1) = n[q(1-q)^3 - q^3(1-q)][\ln(u/d)]^3$$

μ_e, V_e, δ_e = the empirical values of the respective mean, variance, and skewness of the logarithmic return for a period equal in length to n periods
g_1 = logarithmic return for one period

The values of u, d, and q that satisfy this system of equations are:

$$u = e^{\frac{\mu_e}{n} + \left[\frac{V_e(1-q)}{nq}\right]^{1/2}} \tag{4}$$

$$d = e^{\frac{\mu_e}{n} - \left[\frac{V_e q}{n(1-q)}\right]^{1/2}} \tag{5}$$

$$q = \frac{1}{2} \pm \frac{1}{2}\left[\frac{4V_e^3}{n\delta_e^2} + 1\right]^{-1/2}, \quad -\text{ if } \delta_e > 0, + \text{ if } \delta_e < 0 \tag{6}$$

If δ_e is positive (negative), then q is less (greater) than 0.5; if skewness is zero, then q = 0.5 and Equations (4) and (5) simplify to the Cox, Ross, and Rubinstein (CRR) binomial option pricing model formulas for u and d (Cox, et.al., 1979):

$$u = e^{\frac{\mu_e}{n} + \left[\frac{V_e}{n}\right]^{1/2}} \tag{7}$$

$$d = e^{\frac{\mu_e}{n} - \left[\frac{V_e}{n}\right]^{1/2}} \tag{8}$$

In the skewness-adjusted binomial model, the presence of skewness affects the relative contribution of the mean to the values of u and d. In the case of a positive mean, the mean becomes more important in determining the value of u, the greater the negative skewness. By contrast, in the case of a negative mean, the mean becomes more important in determining the value of d, the greater the positive skewness. In addition, the skewness-adjusted model does not have the same asymptotic properties as the normally distributed CRR model. For large n, the CRR model depends only on the variance of the underlying asset. In the skewness model, u and d depend on all three moments for the case of large n.

BOND AND OPTION VALUATION WITH A BINOMIAL INTEREST RATE TREE

If interest rates are expected to increase in the future, then the values of an option-free bond and a call option on the bond should be lower and the value of a put option on the bond should be greater than the value that would occur in cases in which interest rates are expected to be stable or decrease. For an increasing interest rate scenario, an option-free bond and bond derivatives should, in turn, be valued by a binomial model that reflects a positive expected logarithmic return and negative skewness, and based on expectations theory, the implied spot yield curve generated from the binomial tree should be positively sloped to reflect the expectation of higher rates. In contrast, if rates are expected to decrease in the future, then the values of option-free bonds and the values of put options should be lower and the values of a call options on bonds should be greater than the values that would occur when stable or increasing future interest rate patterns are expected. In this case, such bonds and bond derivatives should be valued by a binomial model that reflects a negative expected return and possibly positive skewness, and the implied spot yield curve generated from the binomial tree should be negatively sloped to reflect the expectation of lower rates.

Figure 1 shows the equilibrium model values for a 7.5-year, 10% option-free bond and on-the-money spot American and European call and put options on the bond with expirations of two years (n = 8 quarters). The tree is subdivided into 30 periods of length one quarter, with the bond yielding quarterly accrued interest of \$2.50 and with the initial quarterly spot rate equal to 2.5% (Figure 1 shows the first eight quarters). The bond and spot options are valued for an increasing interest rate scenario in which the distribution at the end of the 7.5 year period is characterized by expected parameter values of μ_e = 0.37018160, V_e = 0.005012997, and δ_e = −0.00007483. The quarterly u and d values that calibrate the 30-period binomial tree to this end of the period distribution are u = 1.02, d = 0.99 and q = 0.75 using Equations (4), (5), and (6):

$$u = e^{\frac{\mu_e}{n} + \left[\frac{V_e(1-q)}{nq}\right]^{1/2}} = e^{\frac{0.37018160}{30} + \left[\frac{0.005012997(1-0.75)}{(30)(0.75)}\right]^{1/2}} = 1.02$$

$$d = e^{\frac{\mu_e}{n} - \left[\frac{V_e q}{n(1-q)}\right]^{1/2}} = e^{\frac{0.37018160}{30} - \left[\frac{0.005012997(0.75)}{(30)(1-0.75)}\right]^{1/2}} = 0.99$$

$$q = \frac{1}{2} \pm \frac{1}{2}\left[\frac{4V_e^3}{n\delta_e^2} + 1\right]^{-1/2} = \frac{1}{2} - \frac{1}{2}\left[\frac{4(0.005012997)^3}{(3)(-0.00007483)^2} + 1\right]^{-1/2} = 0.75$$

The option-free bond's value of $B_0 = 91.39$ is obtained by rolling the 102.50 bond value at maturity to the present with the value at each node equal to the present value of the expected cash flow and with the probability of the increase in one period equal to q = 0.75. The price of the on-the-money European call on the bond, C^e, is \$0.25 (see Figure 1) and the price of the American call, C^a, is 0.40. The European price is obtained by determining the call's intrinsic value at expirations and then rolling the tree to the present where the call price at each node is equal to present value of the expected call value for the next period. The price of the American option is obtained by constraining the price at each node to be the maximum of its intrinsic value or the binomial value. The European put value, P^e, of 1.08 and American value, P^a, of 1.12, in turn, are determined similar to the call options.

The expectation of the increasing interest rate scenario shown in Figure 1 results in a binomial tree in which the bond prices for zero coupon bonds reflect a positively-sloped spot yield curve and higher implied forward rates (see Figure 2). Appendix A shows the binomial bond prices for zero coupon bonds with maturities from one quarter to 30 quarters, the quarterly yields on the implied spot yield curve ($y_{M:}$ $y_{1,}$ $y_{2,}$ … $y_{30:}$), and their implied quarterly forward rates for one year (four quarters), two years (eight quarter), four years (16 quarters), six years (24 quarters), and 7.5 years (30 quarters) from the present. The resulting positively sloped yield curve and higher implied forward rates (see Figure 2 and Appendix A) are, in turn, consistent with an expectation of higher interest rates.

In contrast to the increasing rate case, if the market expects lower rates in the future with the expected distribution of logarithmic returns of spot rates having the following estimated parameters after thirty quarters of $\mu_e = -0.37093163$, $V_e = 0.004979467$, and $\delta_e = 0.00007408$, then the quarterly u and d values that calibrate the 30-period binomial tree to this end of the period distribution would be u = 1.01, d = 0.98, and q = 0.25 using Equations (4), (5), and (6). This tree would price the 7.5-year, 10% option-free bond at $B_0 = 107.75$, and would generate binomial bond prices for zero coupon bonds that would result in a negatively sloped spot yield curve and decreasing implied forward rates, which are consistent with a market expectation of lower rates (Figure 2 and Appendix A). Finally, if the market expects a stable interest environment in which the distribution at the end of the 7.5 year period is characterized by expected parameter values of $\mu_e = 0$, $V_e = 0.011764321$, and $\delta_e = 0$, then the quarterly u and d values that calibrate the 30-period binomial tree to this end-of-the-period distribution would be u = 1.02, d = 1/1.02, and q = 0.5. This tree would yield an option-free bond value of 100, and binomial bond prices for zero coupon bonds that yield a flat spot yield curve and constant implied forward rates (Figure 2 and Appendix A). Table 1 summarizes the equilibrium model values for the 7.5-year, 10% coupon bond and the values for the on-the-money options for the stable, increasing, and decreasing cases.

BINOMIAL FUTURES AND FUTURES OPTIONS PRICES

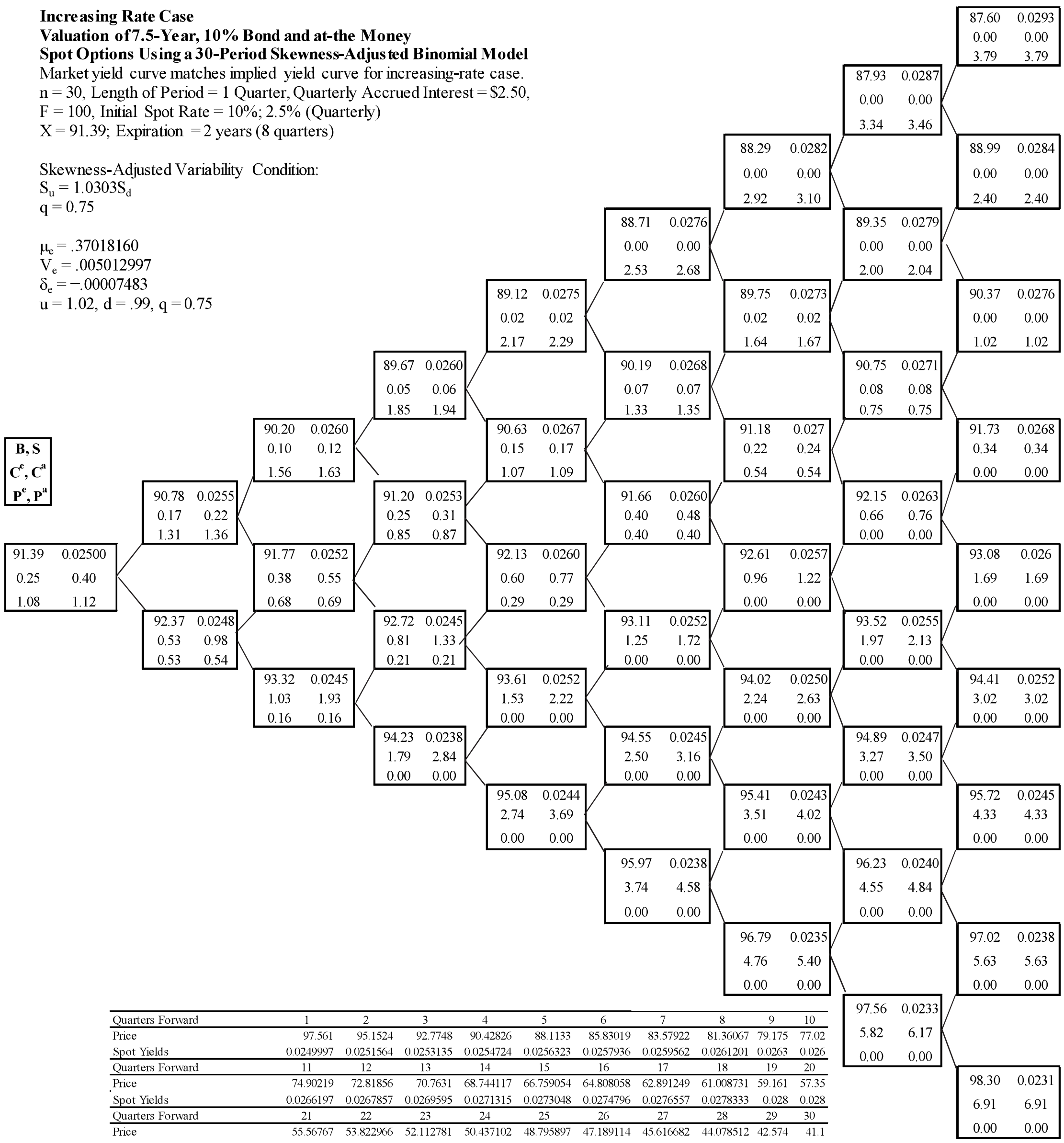
Figure 3 shows the binomial tree values for a futures contract on the 7.5-year, 10% option-free bond, and the values of on-the-money futures options for the decreasing interest rate case (u = 1.01, d = 0.98, and q = 0.25). The futures and futures options have the same expirations of two years (n = 8 quarters). The equilibrium futures price of 108.47 shown in the figure is determined by the carrying-cost model:

$$f_0 = [B_0 - PV(C)](1 + y_{n_f})^{n_f}$$

where PV(C) is the present value of coupons paid on the bond during the life of the futures contract, n_f is the number of periods to the expiration on the futures contract, and y_{nf} = n-period risk-free rate for the expiration period on the futures contract.

FIGURE 1
BINOMIAL TREE—INCREASING CASE VALUATION OF 7.5-YEAR, 10% BOND AND AT-THE MONEY SPOT OPTIONS USING A 30-PERIOD SKEWNESS-ADJUSTED EQUILIBRIUM MODEL AND CALIBRATION MODEL

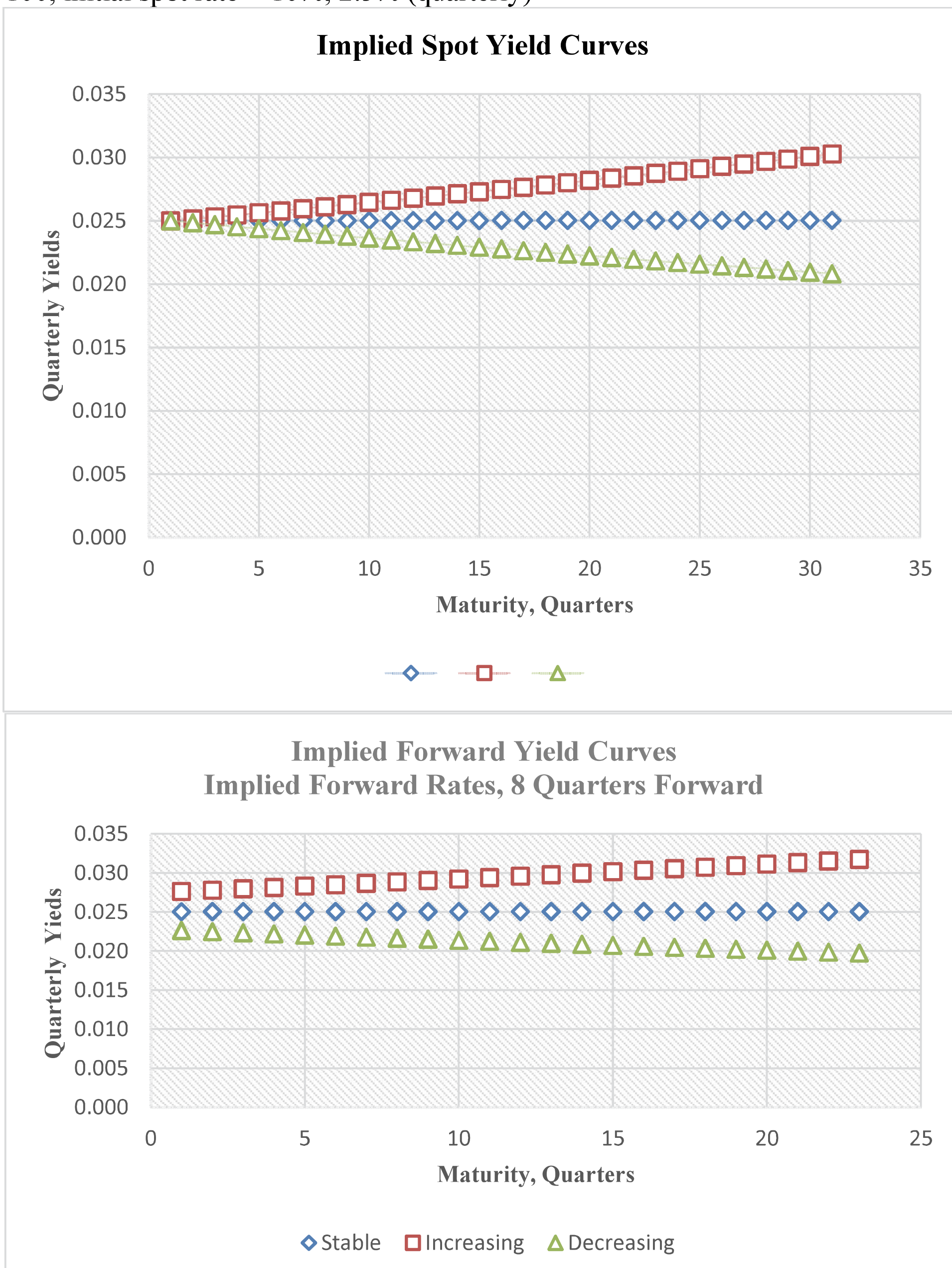
- Equilibrium Model: u = 1.02, d = 0.99, and q =0.75
- Calibration Model: Tree Calibrated to yield curve that matches the implied yield curve from the equilibrium model and the BDT variability conditions are adjusted for skewness adjusted



Quarters Forward	1	2	3	4	5	6	7	8	9	10
Price	97.561	95.1524	92.7748	90.42826	88.1133	85.83019	83.57922	81.36067	79.175	77.02
Spot Yields	0.0249997	0.0251564	0.0253135	0.0254724	0.0256323	0.0257936	0.0259562	0.0261201	0.0263	0.026
Quarters Forward	11	12	13	14	15	16	17	18	19	20
Price	74.90219	72.81856	70.7631	68.744117	66.759054	64.808058	62.891249	61.008731	59.161	57.35
Spot Yields	0.0266197	0.0267857	0.0269595	0.0271315	0.0273048	0.0274796	0.0276557	0.0278333	0.028	0.028
Quarters Forward	21	22	23	24	25	26	27	28	29	30
Price	55.56767	53.822966	52.112781	50.437102	48.795897	47.189114	45.616682	44.078512	42.574	41.1
Spot Yields	0.0283746	0.0285579	0.0287427	0.028929	0.0291168	0.0293061	0.0294969	0.0296893	0.0299	0.03

FIGURE 2
IMPLIED SPOT YIELD CURVES AND IMPLIED FORWARD YIELD CURVES GENERATED FROM THE SKEWNESS-ADJUSTED BINOMIAL MODEL

n = 30, length of period = 1 quarter, quarterly accrued interest = $2.50,
F = 100, initial spot rate = 10%; 2.5% (quarterly)



The binomial futures prices are generated by assuming the market spot yield curve is equal to the implied spot yield curve. The equilibrium futures price ensures that no cash-and-carry arbitrage opportunities exist or equivalently that the total return implied on the futures contracts, TR^f, are equal to the implied forward rate. The implied futures total return on the 10%, 7.5-year bond (2.5% quarterly interest and maturity of 22 quarters) on a futures contract expiring in two years (eight quarters), $TR^f_{22,8}$, is equal to the return obtained from buying the bond at the futures prices, $f_{0,}$ at expiration (quarter 8) and investing the \$2.50 quarterly coupons at implied forward rates starting in quarter nine and going forward to the maturity (quarter 30) and then receiving the \$102.50 principal at maturity:

$$TR^f_{22,8} = \left[\frac{\$2.50(1+y^f_{21,9})^{21} + \$2.50(1+y^f_{20,10})^{20} + \cdots + \$2.50(1+y^f_{1,29})^{1} + \$102.50}{f_0} \right] - 1$$

For the decreasing rate case the current futures price is 108.47 on the contract and the implied total return is 1.986%, which matches the implied forward rate for 22-quarter bond, eight periods forward, $y^f_{22,8}$ (see Appendix A).

The European futures call and put option prices shown in Figure 3 are obtained by determining the options' intrinsic values at expirations and then rolling the tree to the present where the option price at each node is equal to present value of the expected option values for the next period. The price of the American option is obtained by constraining the price at each node to be the maximum of its intrinsic value or the binomial value.

Table 1 summarizes the binomial tree values for a futures contract on the bond and on-the-money futures options for the three cases: increasing interest rate case (u = 1.02, d = 0.99, and q = 0.75), decreasing case (u = 1.01, d = 0.99, q = 0.25), and stable case (u = 1.02, d = 1/1.02, q = 0.5). Comparing spot and futures prices, the futures market for the increasing rate case is inverted in the current period with the futures price less than spot, and after period three it is inverted when rates increase and normal when they decrease. By contrast, the futures market for the decreasing rate case is normal in the current period with the futures price greater than the spot, and after period three it is normal when rates decrease and inverted when they increase. Finally, for the stable rate case the futures price is equal to the spot in the current period, and after period three it is normal when rates decrease and inverted when they increase. In addition, for the stable case the on-the-money European futures call and put options are equal to their respective European spot call and put option prices. However, with futures prices different than spot prices for periods after the current period and prior to maturity, the American futures call is greater than its corresponding American spot call for the stable case.

A SKEWNESS-ADJUSTED CALIBRATION MODEL

A binomial interest rate tree generated using the u and d estimation approach is constrained to have an end-of-the-period distribution with parameter values that match the analyst's estimated value. Since the work of Ho and Lee, it has been widely recognized that binomial interest rate models must possess the no-arbitrage property. The first arbitrage-free binomial interest rate model was the Black, Derman and Toy calibration model. The BDT model is constrained to match the current term structure of spot rates and therefore yields prices for option-free bonds that are equal to their equilibrium values.

The BDT model generates a binomial tree by first finding spot rates that satisfy a variability condition between the upper and lower rates. Given the variability relation, the model then solves for the lower spot rates that satisfies a price condition in which the bond values obtained from the tree are consistent with the equilibrium bond price given the current yield curve of spot rates. The BDT model is therefore constrained to have option-free bond prices that match their equilibrium values. However, the BDT variability condition is not constrained to have an end-of-the-period distribution with parameter values that necessarily match the estimated empirical distribution. Unless the BDT variability conditions are adjusted to reflect the end-of-the-period distribution, calibrating a binomial tree to positively or negatively spot yield curves that reflect increasing or decreasing futures interest rates can lead to mispricing of bond derivatives and opportunities for abnormal returns using the calibration model.

TABLE 1
EQUILIBRIUM MODEL: VALUATION OF 7.5-YEAR, 10% BOND, FUTURES CONTRACT ON THE BOND, AT-THE MONEY SPOT OPTIONS AND FUTURES OPTIONS USING A 30-PERIOD SKEWNESS-ADJUSTED BINOMIAL MODEL

- Bond: n = 30, length of period = 1 quarter, quarterly accrued = \$2.50, f = 100
- Initial spot rate = 10%; 2.5% (quarterly)
- Options: On-the-money, expiration = two years (eight quarters)
- Futures contract on 7.5 year bond, futures expiration = two years; n = 8 quarter
- Futures options expiration = two years; n = 8 quarterly periods; at-the-money futures options

Stable Rate Case: $\mu_e = 0$, $V_e = 0.011764321$, $\delta_e = 0$ u = 1.02, d = 1/1.02, q = 0.5

Option-Free Bond	Option Exercise Price	European Spot Call	American Spot Call	European Spot Put	American Spot Put
100	100	0.75	0.89	0.75	0.89
Futures Price	Futures Option Exercise Price	European Futures Call	American Futures Call	European Futures Put	American Futures Put
100	100	0.75	0.99	0.75	0.89

Increasing Rate Case: $\mu_e = 0.37018160$, $V_e = 0.005012997$, $\delta_e = -0.00007483$, u = 1.02, d = 0.99, q = 0.75

Option-Free Bond	Option Exercise Price	European Call	American Call	European Put	American Put
91.39	91.39	0.25	0.40	1.08	1.12
Futures Price	Futures Option Exercise Price	European Futures Call	American Futures Call	European Futures Put	American Futures Put
90.37	90.37	0.52	0.71	0.52	0.64

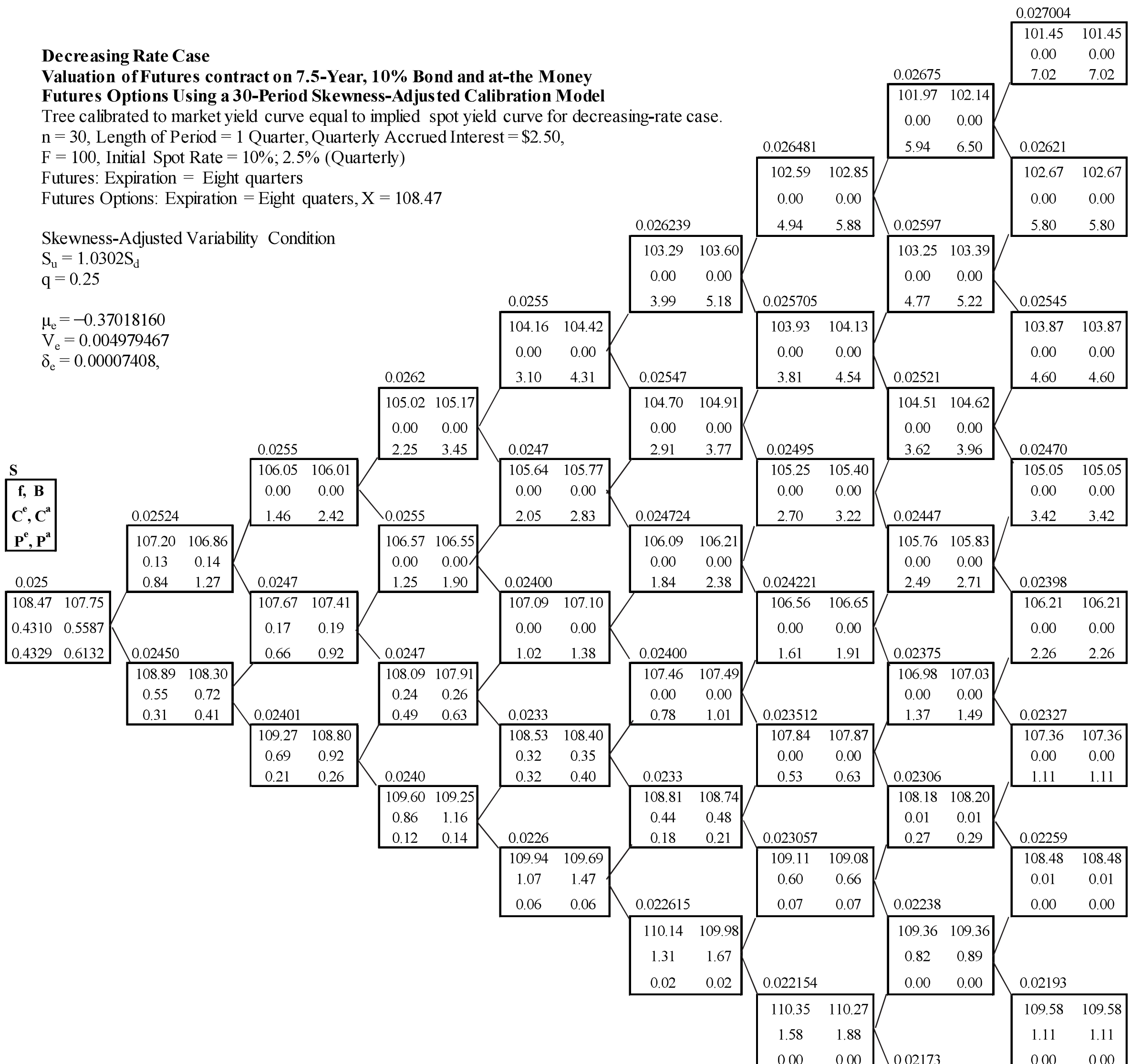
Decreasing Rate Case: $\mu_e = -0.37018160$, $V_e = 0.004979467$, $\delta_e = 0.00007408$, u = 1.01, d = 0.98, q = 0.25

Option-Free Bond	Option Exercise Price	European Call	American Call	European Put	American Put
107.75	107.75	0.84	0.93	0.24	0.37
Futures Price	Option Exercise Price	European Futures Call	American Futures Call	European Futures Put	American Futures Put
108.47	108.47	0.0004	0.1035	3.75	3.75

μ_e, V_e, and δ_e are the mean, variance, and skewness of the spot rate's logarithmic return for a period of 7.5 years. u and d parameters are the upward and downward parameters for a period of one quarter of a year.

FIGURE 3
BINOMIAL TREE—DECREASING RATE CASE: VALUATION OF 7.5-YEAR, 10% BOND, FUTURES, AND AT-THE MONEY FUTURES OPTIONS USING A 30-PERIOD SKEWNESS-ADJUSTED EQUILIBRIUM MODEL AND CALIBRATION MODEL

- Equilibrium model: u = 1.01, d = 0.98, and q =0.25
- Calibration model: Tree calibrated to yield curve that matches the implied yield curve from the equilibrium model and the BDT variability conditions are adjusted for skewness



Quarters Forward	1	2	3	4	5	6	7	8	9	10
Price	97.56098	95.21048	92.94462	90.75971	88.65224	86.61889	84.6565	82.76205	80.93	79.17
Spot Yields	0.025	0.024844	0.024689	0.024535	0.024382	0.024231	0.024081	0.023932	0.024	0.024
Quarters Forward	11	12	13	14	15	16	17	18	19	20
Price	77.45848	75.80859	74.21366	72.67148	71.1799	69.7369	68.34055	66.989	65.68	64.41
Spot Yields	0.023492	0.023348	0.023205	0.023063	0.022923	0.022783	0.022645	0.022507	0.022	0.022
Quarters Forward	21	22	23	24	25	26	27	28	29	30
Price	63.18598	61.99684	60.84447	59.72748	58.64453	57.59435	56.57572	55.58747	54.63	53.7
Spot Yields	0.022102	0.021969	0.021837	0.021706	0.021577	0.021448	0.02132	0.021193	0.021	0.021

Skewness-Adjusted BDT Variability and Price Conditions

In the BDT calibration model, the variability condition governing the upper and lower spot rates for the first period is given as

$$S_u = S_d\, e^{2\sqrt{V_e/n}} \tag{9}$$

The condition originates from the CRR formulas for estimating u and d:

$$\frac{S_u}{u} = S_0 = \frac{S_d}{d}$$

$$S_u = S_d \frac{u}{d}$$

$$S_u = S_d \frac{e^{\sqrt{V_e/n} + \mu_e/n}}{e^{-\sqrt{V_e/n} + \mu_e/n}} = S_d e^{2\sqrt{V_e/n}}$$

The variability condition can be modified to incorporate skewness by defining the relationship between S_u and S_d in terms of the skewness-adjusted parameters (Equations (4) and (5)):

$$S_u = S_d \frac{u}{d} = S_d \frac{e^{\sqrt{\frac{V_e(1-q)}{nq}} + \frac{\mu_e}{n}}}{e^{-\sqrt{\frac{V_e q}{n(1-q)}} + \frac{\mu_e}{n}}}$$

$$S_u = S_d\, e^{\sqrt{\frac{V_e(1\text{-}q)}{nq}} + \sqrt{\frac{V_e q}{n(1-q)}}} \tag{10}$$

$$q = \frac{1}{2} \pm \frac{1}{2}\left[\frac{4V_e^3}{n\delta_e^2} + 1\right]^{-1/2}$$

The price condition for a one-period interest rate tree requires that the lower rate along with the variability condition yield a binomial value for a two-period option-free bond that is equal to the current equilibrium price. Given the one-period tree, the possible spot rates in period 2 can be found by specifying a similar skewness-adjusted variability condition between S_{dd} and S_{ud} and S_{ud} and S_{uu}, and then solving iteratively for the S_{dd} value that yields a binomial value for a three-period bond that is equal to the bond's equilibrium price. The tree's spot rates for subsequent periods are found in a similar way. Once the tree has been calibrated to the current yield curve, it can then be used to value bonds, their embedded options, and their derivatives.

Pricing Differences with and without Skewness Adjustments

The skewness-adjusted equilibrium model and the skewness-adjusted calibration model yield the same binomial interest rate tree when the market spot yield curve matches the implied yield curve from the expected end-of-the period distribution. For example, in the 30-period increasing interest rate case, the annualized logarithmic mean, variance, and skewness were $\mu_e = 0.37018160$, $V_e = 0.005012997$, and $\delta_e =$

−0.00007483. For the equilibrium model, this yielded parameter value of u = 1.02, d = 0.99, and q = 0.75 and an implied positively sloped spot yield curve (Figure 2 and Appendix A; y_{1q} = 2.5%, y_{2q} = 2.5156398%, y_{3q} = 2.53135%, and so on). If the current market spot yield curve were the same as the implied spot yield curve for the increasing case, then the equilibrium model and calibration model would be the same provided the skewness-adjusted variability conditions are used to calibrate the binomial tree to the market yield curve (see Figure 1). Specifically, for the 30-period calibrated tree, the BDT skewness-adjusted variability condition and q value are

$$S_u = S_d e^{\sqrt{\frac{V_e(1-q)}{nq}} + \sqrt{\frac{V_e q}{n(1-q)}}}$$

$$S_u = S_d e^{\sqrt{\frac{0.005012997(1-0.75)}{(30)(0.75)}} + \sqrt{\frac{(0.0005012997)(0.75)}{(30)(1-0.75)}}} = 1.0303 S_d$$

$$q = \frac{1}{2} \pm \frac{1}{2}\left[\frac{4V_e^3}{n\delta_e^2} + 1\right]^{-1/2} = \frac{1}{2} + \frac{1}{2}\left[\frac{4(.005012997)^3}{(3)(-0.00007483)^2} + 1\right]^{-1/2} = 0.75$$

For the first period, the S_d value that equates the binomial-generated price of a two-quarter zero coupon bond (F = \$100) to the equilibrium price ($100/1.0251564^2$ = 95.1524) is 2.48%, with the upper rate satisfying the variability condition being S_u = (1.0303)(2.475063%) = 2.55%. For the tree's second period, the S_{dd} value that equates the binomial price of a three-quarter zero to its equilibrium price ($100/1.02531^3$ = 92.7745) is 2.450146%, with the successive upper rates satisfying the variability conditions being S_{ud} = 2.52% and S_{uu} = 2.60%. The resulting 30-period calibrated binomial tree matches the binomial interest rate tree generated using the equilibrium model in which μ_e = 0.37018160, V_e = 0.005012997, and δ_e = −0.00007483 (u = 1.02, d = 0.99, and q = 0.75). As shown in Figure 1, the equilibrium model and the calibrated model with the skewness adjusted variability condition both price the 30-period, 2.5% quarterly coupon bond at 91.39, a two-year on-the-money European call on the bond at 0.25 (0.40 for an American call), and an on-the-money European put at 1.08 (1.12 for American).

If the current spot yield curve is again equal to the implied spot yield for the increasing rate scenario, but the BDT variability conditions and probability are not adjusted for skewness, then the resulting binomial tree will have comparatively lower interest rates at most of the nodes. In this case, q = 0.5 and the unadjusted variability condition is

$$S_u = S_d e^{\sqrt{\frac{V_e(1-0.5)}{n(.5)}} + \sqrt{\frac{V_e(0.5)}{n(1-0.5)}}}$$

$$S_u = S_d e^{2\sqrt{\frac{V_e}{n}}} S_d = S_d e^{2\sqrt{\frac{0.005012997}{30}}} = 1.026191 S_d$$

The unadjusted variability condition implies an end-of-the period distribution of stable interest rates and a flat yield curve, which is inconsistent with the market yield curve that reflects a market expectation of increasing rates. Figure 4 shows the first eight quarters of the 30-period calibrated binomial interest rate tree without the skewness adjustment variability along with the corresponding bond prices for the 2.5% quarterly coupon bond, and the prices for the spot options on the bond. Given the calibration model's price constraint, the price of the bond is the same as the skewness-adjusted price of 91.39.

However, the bond's prices in subsequent periods differ from the skewness-adjusted case. As a result, derivatives on the bond have different prices than the skewness-adjusted case. Specifically, the two-year European and American call prices on the bond are 0.22 and 0.33, respectively, 12% and 19.51% less than the skewness-adjusted case (0.25 and 0.41). The two-year European and American put prices on the bond are both 1.11, compared to 1.08 (2.78% greater) and 1.12 (1% greater) for the skewness adjusted model.

In the decreasing-rate case, the BDT skewness adjusted-variability conditions and q value are

$$S_u = S_d e^{\sqrt{\frac{V_e(1-q)}{nq}} + \sqrt{\frac{V_e q}{n(1-q)}}}$$

$$S_u = S_d e^{\sqrt{\frac{0.004979467(1-0.25)}{(30)(0.25)}} + \sqrt{\frac{(0.004979467)(0.25)}{(30)(1-0.25)}}} = 1.0302 S_d$$

$$q = \frac{1}{2} \pm \frac{1}{2}\left[\frac{4V_e^3}{n\delta_e^2} + 1\right]^{-1/2} = \frac{1}{2} + \frac{1}{2}\left[\frac{4(.005979467)^3}{(3)(0.00007408)^2} + 1\right]^{-1/2} = 0.25$$

In this case, the price of the 30-period, 2.5% quarterly coupon bond is 107.75, a two-year on-the-money European call on the bond is 0.84 (0.93 for an American call), and an on-the-money European put is 0.24 (0.37 for American). By contrast, if the variability condition is not adjusted, then q = 0.5 and

$$S_u = S_d e^{2\sqrt{V_e/n}} = S_d e^{2\sqrt{0.004979467/30}} = 1.02619 S_d$$

Given the price constraint, the price of the bond is the same as the skewness-adjusted price of 107.75. However, the bond's prices in subsequent periods differ from the skewness-adjusted case, resulting in different derivative prices than the skewness-adjusted case. For example, the two-year European and American call prices on the bond are 0.91 and 1.01, respectively, compared to 0.84 and 0.93 for the adjusted case. The two-year European and American put prices on the bond are both 0.16 and 0.24, compared to 0.24 and 0.37 for the skewness adjusted model.

In the stable interest case, there is no need to adjust the variability conditions for skewness: q = 0.5 and the variability condition is

$$S_u = S_d e^{2\sqrt{\frac{V_e}{n}}} = S_d e^{2\sqrt{\frac{0.011764321}{30}}} = 1.0404 S_d$$

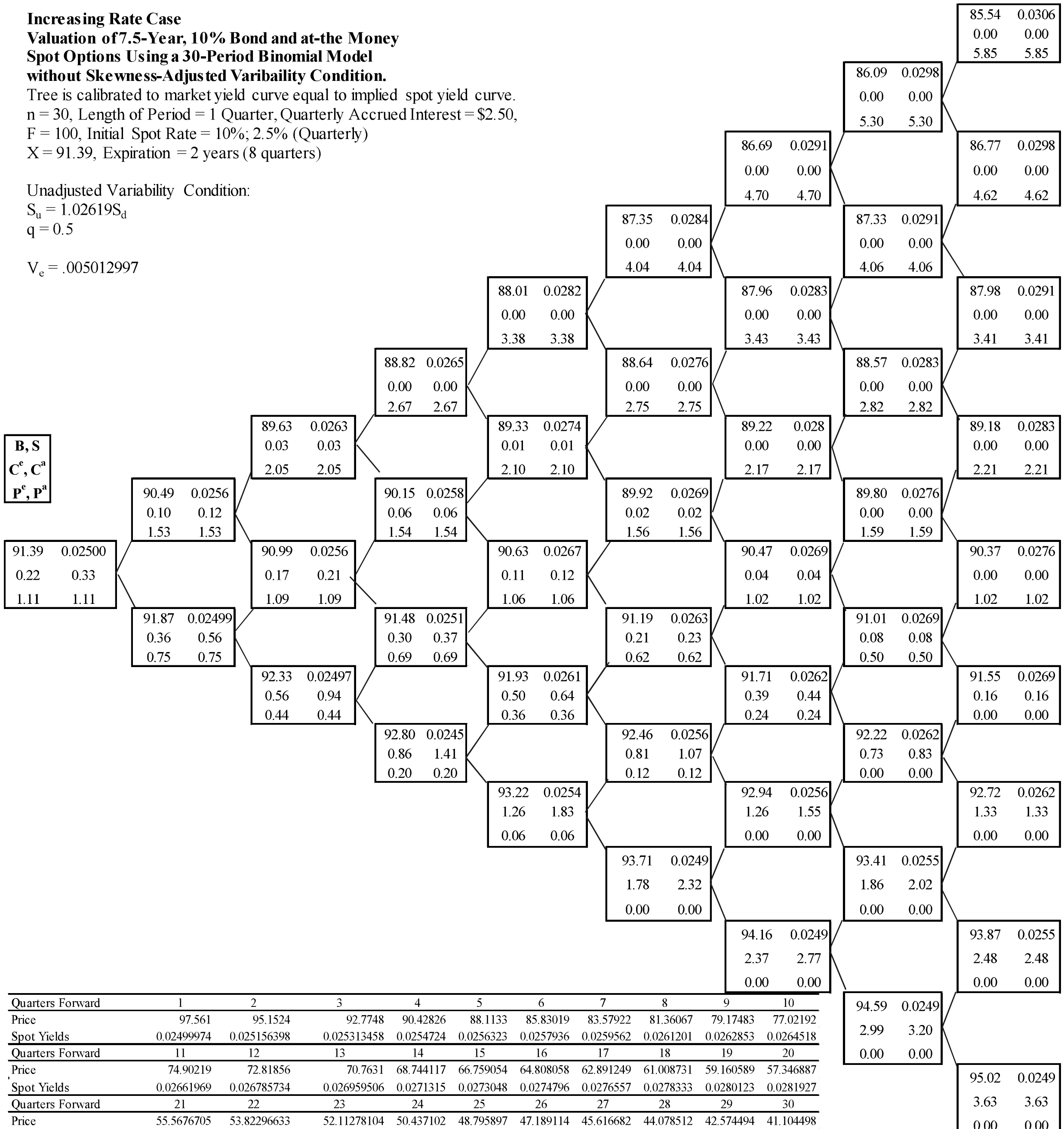
In this case, the derivative prices are the same. It should be noted that in a stable interest case in which the expected mean and skewness are zero, the implied yield curve is flat. If the market yield curve matches the flat implied yield curve, then the calibrated model will generate the same bond, futures, and derivative prices as the equilibrium model. Bond and option prices for the increasing, decreasing, and stable cases with and without the skewness-adjusted variability condition are summarized in Appendix B.

FIGURE 4
30-PERIOD CALIBRATED TREE WITHOUT SKEWNESS ADJUSTMENT: INCREASING RATE CASE

Increasing Rate Case
Valuation of 7.5-Year, 10% Bond and at-the Money Spot Options Using a 30-Period Binomial Model without Skewness-Adjusted Varibaility Condition.
Tree is calibrated to market yield curve equal to implied spot yield curve.
n = 30, Length of Period = 1 Quarter, Quarterly Accrued Interest = $2.50,
F = 100, Initial Spot Rate = 10%; 2.5% (Quarterly)
X = 91.39, Expiration = 2 years (8 quarters)

Unadjusted Variability Condition:
$S_u = 1.02619 S_d$
q = 0.5

$V_e = .005012997$



Quarters Forward	1	2	3	4	5	6	7	8	9	10
Price	97.561	95.1524	92.7748	90.42826	88.1133	85.83019	83.57922	81.36067	79.17483	77.02192
Spot Yields	0.02499974	0.025156398	0.025313458	0.0254724	0.0256323	0.0257936	0.0259562	0.0261201	0.0262853	0.0264518
Quarters Forward	11	12	13	14	15	16	17	18	19	20
Price	74.90219	72.81856	70.7631	68.744117	66.759054	64.808058	62.891249	61.008731	59.160589	57.346887
Spot Yields	0.02661969	0.026785734	0.026959506	0.0271315	0.0273048	0.0274796	0.0276557	0.0278333	0.0280123	0.0281927
Quarters Forward	21	22	23	24	25	26	27	28	29	30
Price	55.5676705	53.82296633	52.11278104	50.437102	48.795897	47.189114	45.616682	44.078512	42.574494	41.104498
Spot Yields	0.02837456	0.0285579	0.028742712	0.028929	0.0291168	0.0293061	0.0294969	0.0296893	0.0298831	0.0300786

Futures and Futures Options

Futures prices and at-the-money futures option prices for the three interest rate scenarios with and without the skewness adjusted variability conditions are also summarizes in Appendix B. As with the spot prices, the futures prices in the current period are the same with and without the skewness adjustment for each scenario, but differ in subsequent periods for the increasing and decreasing cases. As a result, for the decreasing and increasing cases, derivative prices differ. For example, in the decreasing rate case, the futures price is 108.47, the European and American call prices for the skewness-adjusted variability condition are 0.4310 and 0.5587, compared to European and American call prices for the unadjusted condition of 0.5291 and 0.7060. Similar differences also exist for the puts. Figure 3 shows the first eight quarters of the 30-period binomial interest rate tree calibrated to the implied yield curve for the decreasing rate case with the skewness adjustment variability condition (this is also the tree for the skewness-adjusted equilibrium model), and Figure 5 shows the calibrated tree without the skewness-adjustment to the variability condition.

ARBITRAGE IMPLICATIONS

The motivation behind the derivation of the BDT model was to ensure that there is no arbitrage opportunity from buying a bond and stripping it or buying zeros and bundling them. In addition to satisfying an arbitrage-free condition on option-free bonds, the calibration model also ensured that the options on the bond were arbitrage-free. For example, the current European call price of 0.25 for the increasing case (Figure 1) is equal to the value of a portfolio consisting of a one-quarter zero-coupon bond and a two-quarter zero constructed so that next year the portfolio is worth $C_u = 0.17$ if the spot rate is 2.55% and 0.53 if the rate is at 2.48%. Specifically, given the possible cash flows on the two-quarter zero of $B_u = 100/1.0255 = 97.5134$ and $B_d = 109/1.0248 = 97.5800156$, and the cash flow on the one-quarter bond of 100, the replicating portfolio is formed by solving for the number of one-quarter bonds, n_1, and the number of two-quarter bonds, n_2, where:

$$n_1(100) + n_2(97.5134) = 0.17$$

$$n_1(100) + n_2(97.5800156) = 0.53$$

Solving the equations for n_1 and n_2, $n_1 = -5.2687$ and $n_2 = 5.4048$. Thus, a portfolio formed by buying 5.4048 issues of a two-quarter bond and shorting 5.2687 issues of a one-quarter bond will yield possible cash flows next year of 0.17 if the spot rate is at 2.55% and 0.53 if the rate is at 2.48%. Moreover, given the current one-quarter and two-quarter bond prices of 97.56098 (spot yield = y_{1q} = 2.5%) and 95.15068 (spot yield = y_{2q} = 2.5156398%), the value of this replicating portfolio is 0.25:

$$B_1 = \frac{100}{1.025} = 97.56098$$

$$B_2 = \frac{100}{(1.025165398)^2} = 95.15068$$

$$n_1 B_1 + n_2 B_2 = C_0$$

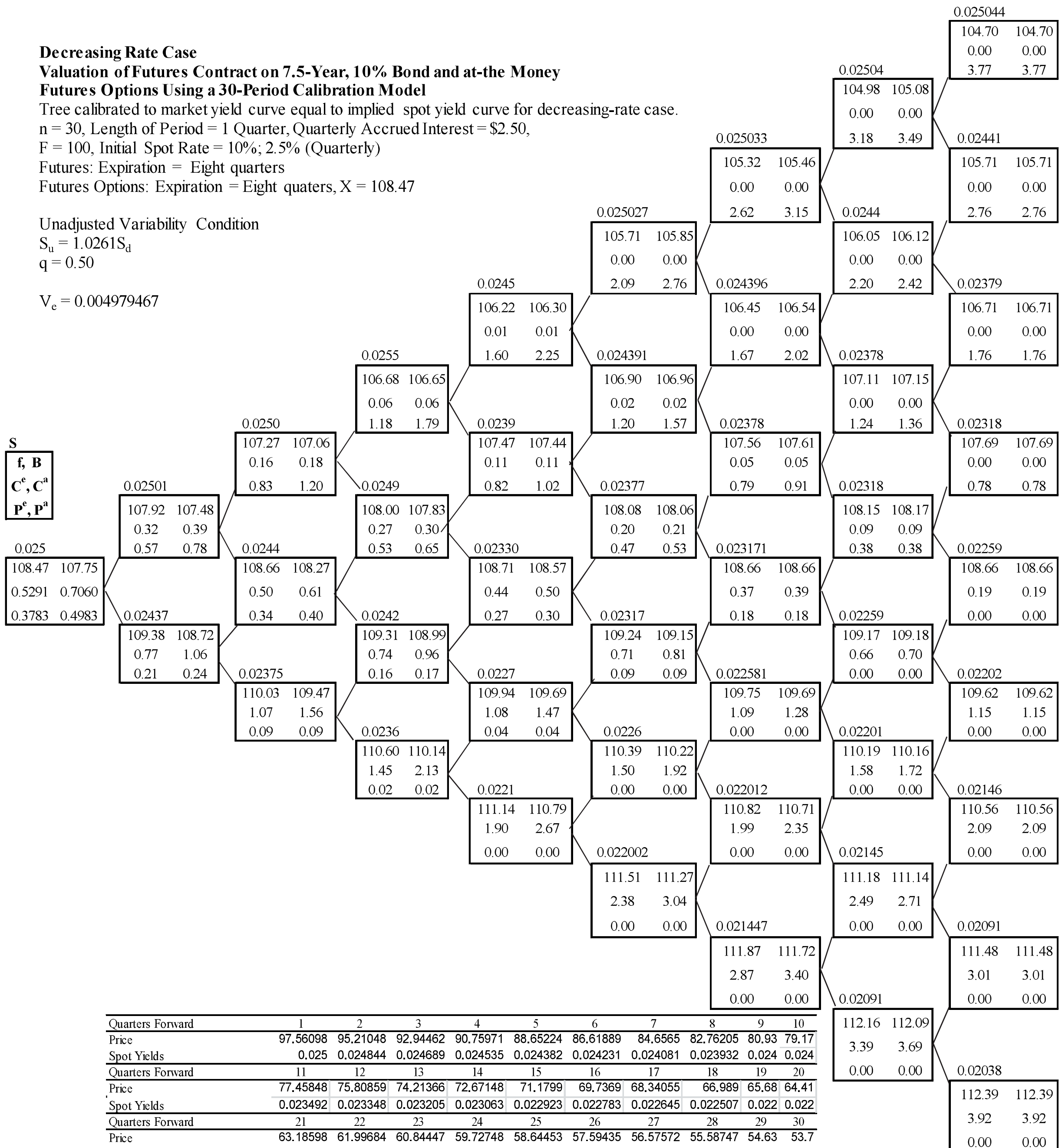
$$(-5.2687)(97.56098) + (5.4048)(95.15173) = 0.25$$

FIGURE 5
SPOT, FUTURES, AND FUTURES OPTIONS PRICES FOR 30-PERIOD CALIBRATED TREE WITHOUT SKEWNESS ADJUSTMENT: DECREASING RATE CASE

Decreasing Rate Case
Valuation of Futures Contract on 7.5-Year, 10% Bond and at-the Money Futures Options Using a 30-Period Calibration Model
Tree calibrated to market yield curve equal to implied spot yield curve for decreasing-rate case.
n = 30, Length of Period = 1 Quarter, Quarterly Accrued Interest = $2.50,
F = 100, Initial Spot Rate = 10%; 2.5% (Quarterly)
Futures: Expiration = Eight quarters
Futures Options: Expiration = Eight quaters, X = 108.47

Unadjusted Variability Condition
$S_u = 1.0261 S_d$
q = 0.50

$V_e = 0.004979467$



Quarters Forward	1	2	3	4	5	6	7	8	9	10
Price	97.56098	95.21048	92.94462	90.75971	88.65224	86.61889	84.6565	82.76205	80.93	79.17
Spot Yields	0.025	0.024844	0.024689	0.024535	0.024382	0.024231	0.024081	0.023932	0.024	0.024
Quarters Forward	11	12	13	14	15	16	17	18	19	20
Price	77.45848	75.80859	74.21366	72.67148	71.1799	69.7369	68.34055	66.989	65.68	64.41
Spot Yields	0.023492	0.023348	0.023205	0.023063	0.022923	0.022783	0.022645	0.022507	0.022	0.022
Quarters Forward	21	22	23	24	25	26	27	28	29	30
Price	63.18598	61.99684	60.84447	59.72748	58.64453	57.59435	56.57572	55.58747	54.63	53.7
Spot Yields	0.022102	0.021969	0.021837	0.021706	0.021577	0.021448	0.02132	0.021193	0.021	0.021

Since the replicating portfolio and the call option have the same cash flow, by the law of one price they must be equally priced. Thus, in the absence of arbitrage, the price of the call is equal to 0.25, which is the same price we obtained by using the skewness-adjusted calibrated model. However, if the variability conditions are not adjusted for skewness, then the price of the European call is 12% underpriced at 0.22 (Figure 4). In this case, an arbitrage exists by buying the call and shorting the replicating portfolio.

Unless constrained otherwise, the BDT model does not require that there be consistency between the spot yield curve on which the tree is being calibrated and the flat yield curve implied by the variability condition without the skewness adjustment. This inconsistency results in mispricing of spot and futures option and an arbitrage when the variability condition is not adjusted to reflect the skewness implied by the end-of-period distribution.

SUMMARY

The binomial interest rate model has become a useful model for determining the values of bonds and bond derivatives, as well as the duration, convexity, and option adjusted spreads of fixed-income instruments with embedded option features. In this paper, we showed how the JPM skewness-adjusted model can be used to calibrate a binomial tree to an increasing-rate scenario in which the end-of-the period distribution is characterized by a positive expected logarithmic return and negative skewness and to decreasing rate case that is characterized by a distribution with a negative expected logarithmic return and positive skewness. We then showed how the implied yield curves and implied forward rates generated from the equilibrium model resulted in yield curves that were consistent with expected end-of-the-period distributions. Finally, we concluded the paper by showing how skewness can be incorporated into the BDT calibration model and showing with a numerical simulation the possible mispricing that can result when the BDT variability conditions are not adjusted to reflect skewness.

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APPENDIX A

TABLE A.1
IMPLIED SPOT YIELD CURVES FROM 30-PERIOD SKEWNESS-ADJUSTED BINOMIAL MODEL

n = 30, length of period = 1 quarter, quarterly accrued interest = $2.50,
F = 100, initial spot rate = 10%; 2.5% (quarterly)

Stable

Maturity Quarters	Binomial Spot Price	Spot Yield	Implied Forward Rates: Quarters (Q) Forward 4 Q	8 Q	16 Q	24 Q	30 Q
1	97.5610	0.025	0.025	0.025	0.025	0.025	0.025
2	95.1810	0.025	0.025	0.025	0.025	0.025	
3	92.8587	0.025	0.025	0.025	0.025	0.025	
4	90.5927	0.025	0.025	0.025	0.025	0.025	
5	88.3816	0.025	0.025	0.025	0.025	0.025	
6	86.2242	0.025	0.025	0.025	0.025	0.025	
7	84.1192	0.025	0.025	0.025	0.025	0.025	
8	82.0653	0.025	0.025	0.025	0.025		
9	80.0613	0.025	0.025	0.025	0.025		
10	78.1060	0.025	0.025	0.025	0.025		
11	76.1983	0.025	0.025	0.025	0.025		
12	74.3371	0.025	0.025	0.025	0.025		
13	72.5211	0.025	0.025	0.025	0.025		
14	70.7494	0.025	0.025	0.025	0.025		
15	69.0209	0.025	0.025	0.025	0.025		
16	67.3345	0.025	0.025	0.025			
17	65.6893	0.025	0.025	0.025			
18	64.0842	0.025	0.025	0.025			
19	62.5183	0.025	0.025	0.025			
20	60.9906	0.025	0.025	0.025			
21	59.5003	0.025	0.025	0.025			
22	58.0463	0.025	0.025	0.025			
23	56.6280	0.025	0.025	0.025			
24	55.2443	0.025	0.025				
25	53.8945	0.025	0.025				
26	52.5777	0.025	0.025				
27	51.2931	0.025	0.025				
28	50.0401	0.025					
29	48.8177	0.025					
30	47.6252	0.025					
31	46.4620	0.025					
Future Terminal Value from reinvesting 2.5 quarterly coupons at implied forward rates starting in 9 Q plus 102.5 at Q 30 =				172.19			
Futures Price =				99.929			
Implied Total Return on futures =				0.025			

Increasing

Maturity Quarters	Binomial Spot Price	Spot Yield	Implied Forward Rates: Quarters (Q) Forward 4 Q	8 Q	16 Q	24 Q	30 Q
1	97.5610	0.02500	0.02627	0.02761	0.03048	0.03363	0.03620
2	95.1524	0.02516	0.02644	0.02778	0.03067	0.03384	
3	92.7748	0.02531	0.02660	0.02795	0.03086	0.03405	
4	90.4283	0.02547	0.02677	0.02812	0.03105	0.03426	
5	88.1133	0.02563	0.02694	0.02830	0.03124	0.03448	
6	85.8302	0.02579	0.02711	0.02848	0.03144	0.03469	
7	83.5792	0.02596	0.02728	0.02866	0.03164	0.03491	
8	81.3607	0.02612	0.02744	0.02884	0.03183		
9	79.1748	0.02629	0.02762	0.02902	0.03203		
10	77.0219	0.02645	0.02780	0.02921	0.03224		
11	74.9022	0.02662	0.02797	0.02939	0.03244		
12	72.8186	0.02679	0.02815	0.02958	0.03264		
13	70.7631	0.02696	0.02833	0.02976	0.03285		
14	68.7441	0.02713	0.02851	0.02995	0.03306		
15	66.7591	0.02730	0.02869	0.03014	0.03327		
16	64.8081	0.02748	0.02887	0.03034			
17	62.8912	0.02766	0.02906	0.03053			
18	61.0087	0.02783	0.02924	0.03073			
19	59.1606	0.02801	0.02943	0.03092			
20	57.3469	0.02819	0.02962	0.03112			
21	55.5677	0.02837	0.02981	0.03132			
22	53.8230	0.02856	0.03000	0.03152			
23	52.1128	0.02874	0.03020	0.03172			
24	50.4371	0.02893	0.03039				
25	48.7959	0.02912	0.03059				
26	47.1891	0.02931	0.03079				
27	45.6167	0.02950	0.03099				
28	44.0785	0.02969					
29	42.5745	0.02988					
30	41.1045	0.03008					
31	39.6684	0.03028					
Future Terminal Value from reinvesting 2.5 quarterly coupons at implied forward rates starting in 9 Q plus 102.5 at Q 30 =				178.866			
Futures Price =				90.36541			
Implied Total Return on futures =				0.03152			

Decreasing

Maturity Quarters	Binomial Spot Price	Spot Yield	Implied Forward Rates: Quarters (Q) Forward 4 Q	8 Q	16 Q	24 Q	30 Q
1	97.5610	0.02500	0.02377	0.02260	0.02043	0.01847	0.01712
2	95.2105	0.02484	0.02362	0.02246	0.02030	0.01835	
3	92.9446	0.02469	0.02348	0.02232	0.02018	0.01824	
4	90.7597	0.02453	0.02333	0.02218	0.02005	0.01812	
5	88.6522	0.02438	0.02318	0.02204	0.01993	0.01801	
6	86.6189	0.02423	0.02304	0.02191	0.01980	0.01790	
7	84.6565	0.02408	0.02290	0.02177	0.01968	0.01778	
8	82.7620	0.02393	0.02276	0.02164	0.01956		
9	80.9327	0.02378	0.02261	0.02150	0.01943		
10	79.1657	0.02364	0.02248	0.02137	0.01931		
11	77.4585	0.02349	0.02234	0.02124	0.01920		
12	75.8086	0.02335	0.02220	0.02111	0.01908		
13	74.2137	0.02321	0.02206	0.02098	0.01896		
14	72.6715	0.02306	0.02193	0.02085	0.01884		
15	71.1799	0.02292	0.02180	0.02072	0.01873		
16	69.7369	0.02278	0.02166	0.02060			
17	68.3405	0.02264	0.02153	0.02047			
18	66.9890	0.02251	0.02140	0.02035			
19	65.6805	0.02237	0.02127	0.02022			
20	64.4134	0.02224	0.02114	0.02010			
21	63.1860	0.02210	0.02101	0.01998			
22	61.9968	0.02197	0.02089	0.01986			
23	60.8445	0.02184	0.02076	0.01974			
24	59.7275	0.02171	0.02064				
25	58.6445	0.02158	0.02051				
26	57.5943	0.02145	0.02039				
27	56.5757	0.02132	0.02027				
28	55.5875	0.02119					
29	54.6285	0.02107					
30	53.6977	0.02094					
31	52.7941	0.02082					
Future Terminal Value from reinvesting 2.5 quarterly coupons at implied forward rates starting in 9 Q plus 102.5 at Q 30 =				167.4644			
Futures Price =				108.6543			
Implied Total Return on futures =				0.01986			

Furutes on 2.5 coupon bond with 22 quarter maturity and with futures expiration at 8 Q

APPENDIX B

TABLE B.1
VALUATION OF 7.5-YEAR, 10% BOND AND AT-THE MONEY SPOT AND FUTURES OPTIONS USING A 30-PERIOD CALIBRATION MODEL WITH AND WITHOUT SKEWNESS-ADJUSTED VARIABILITY CONDITIONS

- Tree calibrated to market yield curves equal to the Equilibrium Model's implied spot yield curves
- n = 30, Length of Period = 1 Quarter, Quarterly Accrued Interest = $2.50
- F = 100, Initial Spot Rate = 10%; 2.5% (Quarterly); Spot Options: On-the-money, Expiration = two years (eight quarters)
- Futures Contract on 7.5-year bond, Futures Expiration = two years; n = 8 quarter
- Futures Options Expiration = two years; n = 8 quarterly periods; On-the-money futures options

Increasing Rate Case: $\mu_e = 0.37018160$, $V_e = 0.005012997$, $\delta_e = -0.00007483$
u = 1.02, d = 0.99, q = 0.75, Calibrated to Equilibrium Model Implied Yield Curve for Increasing Case (Exhibit 4 Spot yields Increasing Case)

- Skewness-Adjusted BDT Variability Condition: $S_u = 1.0303S_d$; q = 0.75
- Unadjusted BDT Variability Condition: $S_u = 1.02619S_d$; q = 0.5

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Spot Option Exercise Price	Skewness-Adjusted European Call Price	Unadjusted European Call Price	Skewness-Adjusted American Call Price	Unadjusted American Call Price
91.39	91.39	91.39	0.25	0.22	0.40	0.33
Skewness-Adjusted Bond Price	**Unadjusted Bond Price**	**Spot Option Exercise Price**	**Skewness-Adjusted European Put Price**	**Unadjusted European Put Price**	**Skewness-Adjusted American Put Price**	**Unadjusted American Put Price**
91.39	91.39	91.39	1.08	1.11	1.12	1.11

Futures and Futures Options

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Futures Price	Futures Exercise Price	Skewness-Adjusted European Futures Call Price	Unadjusted European Futures Call Price	Skewness-Adjusted American Futures Call Price	Unadjusted American Futures Call Price
91.39	91.39	90.37	90.37	0.52	0.53	0.71	0.68
Skewness-Adjusted Bond Price	**Unadjusted Bond Price**	**Futures Price**	**Futures Exercise Price**	**Skewness-Adjusted European Futures Put Price**	**Unadjusted European Futures Put Price**	**Skewness-Adjusted American Futures Put Price**	**Unadjusted American Futures Put Price**
91.39	91.39	90.37	90.37	0.52	0.53	0.64	0.68

TABLE B.1, Continued

Decreasing Rate Case: $\mu_e = -0.37018160$, $V_e = 0.004979467$, $\delta_e = 0.00007408$
u = 1.01, d = 0.98, q = 0.25, Calibrated to Equilibrium Model Implied Yield Curve for Decreasing Case (Exhibit 4 Spot yields Decreasing Case)

- Skewness-Adjusted BDT Variability Condition: $S_u = 1.0302S_d$; q = 0.25
- Unadjusted BDT Variability Condition: $S_u = 1.0261$; S_d; q = 0.5

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Option Exercise Price	Skewness-Adjusted European Call Price	Unadjusted European Call Price	Skewness-Adjusted American Call Price	Unadjusted American Call Price
107.75	107.75	107.75	0.84	0.91	0.93	1.01
Skewness-Adjusted Bond Price	**Unadjusted Bond Price**	**Option Exercise Price**	**Skewness-Adjusted European Put Price**	**Unadjusted European Put Price**	**Skewness-Adjusted American Put Price**	**Unadjusted American Put Price**
107.75	107.75	107.75	0.24	0.16	0.37	0.24

Futures and Futures Options

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Futures Price	Futures Exercise Price	Skewness-Adjusted European Call Price	Unadjusted European Futures Call Price	Skewness-Adjusted American Futures Call Price	Unadjusted American Futures Call Price
107.75	108.47	108.47	108.47	0.4310	0.5291	0.5587	0.7060

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Futures Price	Futures Exercise Price	Skewness-Adjusted European Put Price	Unadjusted European Futures Put Price	Skewness-Adjusted American Futures Put Price	Unadjusted American Futures Put Price
107.75	108.47	108.47	108.47	0.4329	0.3783	0.6132	0.4983

Stable Rate Case: $\mu_e = 0$, $V_e = 0.011764321$, $\delta_e = 0$
u = 1.02, d = 1/1.02, q = 0.5; Calibrated to Equilibrium Model Implied Yield Curve for Stable Case (Exhibit 4 Spot yields Decreasing Case)

- Skewness-Adjusted BDT Variability Condition: $S_u = 1.0404S_d$; q = 0.5
- Unadjusted BDT Variability Condition: $S_u = 1.0404$; S_d; q = 0.5

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Option Exercise Price	Skewness-Adjusted European Call Price	Unadjusted European Call Price	Skewness-Adjusted American Call Price	Unadjusted American Call Price
100	100	100	0.75	0.75	0.89	0.89
Skewness-Adjusted Bond Price	**Unadjusted Bond Price**	**Option Exercise Price**	**Skewness-Adjusted European Put Price**	**Unadjusted European Put Price**	**Skewness-Adjusted American Put Price**	**Unadjusted American Put Price**
100	100	100	0.75	0.75	0.89	0.89

Futures and Futures Options

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Futures Price	Futures Exercise Price	Skewness-Adjusted European Call Price	Unadjusted European Futures Call Price	Skewness-Adjusted American Futures Call Price	Unadjusted American Futures Call Price
100	100	100	100	0.75	0.75	0.99	0.99

Skewness-Adjusted Bond Price	Unadjusted Bond Price	Futures Price	Futures Exercise Price	Skewness-Adjusted European Put Price	Unadjusted European Futures Put Price	Skewness-Adjusted American Futures Put Price	Unadjusted American Futures Put Price
100	100	100	100	0.75	0.75	0.99	0.99