A Framework for Minimizing the Tracking Error in an Indexed Portfolio Through Efficient Tax Management

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This paper presents a framework for an investor to minimize a loss function that includes the total costs from tracking errors, capital tax losses, and transaction costs. Using this framework, we analyze optimal trading decisions and suggest a trading rule called the "x-percent rule," which minimizes the loss function. According to this trading rule, once the price of a stock position in the portfolio drops x percent from its purchase price, the portfolio manager sells that position and reinvests in another stock from the same sector. Numerically, the proposed framework is applied to simulated asset returns based on parameters calibrated from historical U.S. stock market returns.

INTRODUCTION

Empirical evidence indicates that actively managed mutual funds do not earn abnormal returns and do not beat the market. On average, active managers cannot improve upon the returns derived from passive investment strategies (Frino and Gallagher, 2001; Sharpe, 1991). Over time, there has been significant growth in both the proportion of assets invested in index portfolios and the number of index mutual funds. Funds tracking broad U.S. stock indexes had more assets by value than stock-picking rivals for the first time in 2019 (Lawn, 2019). As a result, the importance of index funds has significantly increased.

A central issue facing index funds is how to assess their performance. Portfolio performance is usually evaluated relative to a specific benchmark by measuring the tracking error (TE), which "refers to the extent to which a portfolio deviates from its intended behavior." Quantitatively, the TE is the difference between an index portfolio's and the benchmark portfolio's returns. Index fund managers strive to minimize a portfolio's TE while maintaining comparable risk.

There is extensive literature that focuses on various ways to minimize TE. For example, Jansen and van Dijk (2002) illustrate optimization techniques to minimize TE, including a method for choosing the right stocks in the tracking portfolio. Rudolf et al. (1999) investigate four models for minimizing TE. In their modeling, they examine four alternative definitions to the mean squared error: mean absolute deviation (MAD), minimized maximum deviation (MinMax), mean absolute downside deviation (MADD), and downside MinMax (DMinMax). Hwang and Satchell (2016) suggest a source of bias in TE, which arises from the stochastic nature of portfolio weights. Ammann and Tobler (2000) introduce a factor model and study the decomposition of TE variance and sampling error.

One source of TE is transaction costs. A manager's attempt to minimize transaction costs can affect the initial allocations to different asset classes, the portfolio rebalancing decisions, and the frequency of trading. Pliska and Suzuki (2004) study the asset allocation problem of optimally tracking a target mix of asset categories when there are transaction costs. They consider the trading strategy for an investor who is trying to minimize both the fixed and the proportional transaction costs while simultaneously minimizing the TE for a specified target asset mix.

In another paper, Barro and Canestrelli (2009) are interested in dynamically replicating a benchmark portfolio using only a small subset of assets, accounting for transaction costs due to rebalancing, and introducing a liquidity component to the portfolio. Kissell et al. (2004) also provide a decision-making framework that estimates transaction costs. The methodology they develop can adapt to strategies aimed at preserving asset values, achieving the closing price or volume-weighted average price ("VWAP"), and minimizing TE. Optimal portfolio strategy or optimal trading strategy papers without benchmarking consider transaction costs more often in their analysis compared to index-tracking papers. Moorman (2014) summarizes the various transaction costs confronting portfolio managers and searches for methods to reduce them.

The literature on portfolio tracking errors analyzes optimization techniques for TE minimization, with special attention to the transaction costs associated with the frequency of trading. While taxes are an important factor that affects managers' portfolio decisions, there is scant research on ways to reduce the tax impact on index portfolio returns, when considering transaction costs. Past researchers have shown interest in discovering ways to reduce capital gains taxes or increase capital tax losses (Huang, 2008). Moreover, past research has verified that mutual fund investors are interested in receiving tax-management advice from their financial advisors, along with other financial services (Cici, Kempf, and Sorhage, 2017). Dammon and Spatt (1996) and Constantinides (1984) are two pioneering papers suggesting that selling the tax losses could be an optimal way to use the capital losses, but their focus is not the capital tax losses of an index portfolio. Sialm and Sosner (2018) find that the performance of U.S. stock mutual funds is related to their tax burdens, and tax-efficient funds exhibit superior after-tax performance. In another recent paper, Shomesh et al. (2020) show that a tax-loss-harvesting strategy yields a before-transaction-cost tax alpha of 1.08% per year for the 500 securities with the largest market capitalizations from 1926 to 2018. Furthermore, Goldberg et al. (2019) also examine the tax efficiency of an indexing strategy and show that, between June 1995 and March 2018, the average value added by focused attention to tax management exceeded 1.50% per year at a 10-year horizon for all the strategies considered.

Although there is extensive literature on tracking error, our review of the literature finds little on tax-management and transaction costs when benchmarking. Joint analysis of tracking errors, transaction costs, and tax losses have not received a great deal of attention in the literature to date. In this paper, we consider a strategy for a portfolio manager who is trying to minimize the impact from taxes, while simultaneously minimizing the TE and transaction costs. In the analysis presented in this paper, defining the loss function as the total costs from tracking errors, transaction costs, and tax losses, we numerically show the convexity of the loss function, the existence of an optimal strategy, and compute it numerically.

Our paper differs from previous studies and contributes to the literature in several important ways. First, we present a practical framework for a portfolio manager with a three-fold objective function: Minimize the tracking error by minimizing taxes and transaction costs. Second, we derive an analytical solution for portfolio weights and selection of stocks as a function of market, industry, and individual stock volatilities that will keep the TE minimized for a tax-managed indexed portfolio. And third, we introduce a strategy that we term the "*x*-percent rule," which defines optimal timing for tax-loss harvesting based on the analytical solution and the three-fold objective function. Finally, considering alternative scenarios and parameters in simulation analysis, we numerically show the convexity of the loss function with the three-fold objective and the existence of an optimal *x*-percent rule strategy.

Theoretical and completely analytical solutions for complex but practical portfolio optimization problems are not feasible. To assess and evaluate alternative scenarios, the proposed framework is applied to simulated asset returns whose parameters are calibrated from historical U.S. stock market returns. Simulation methods are popular in the portfolio literature (Bajeux-Besnainou et al., 2011; Leland, 2013) to study optimization and trading strategies. When analyzing alternative scenarios for parameters and assumptions, they find that portfolio managers should sell stocks after they incur a 7% to 12% decline in price to generate tax losses optimally, depending on the stocks' returns and return volatilities computed from historical stock prices, investment horizons, tax rates, transaction cost parameters, and tracking deviation penalties.

METHODOLOGY

We derive and illustrate an index approach that minimizes tracking errors, by lowering transaction costs and efficient use of tax losses. We contribute to the TE literature by introducing an approach that incorporates the stock-, industry-, and market-specific volatilities. This section presents the assumptions, definitions, and technical details of the methodology used.

Assumptions

Throughout our analysis, we assume a total of N firms in the benchmark used for index portfolios. These firms are divided into n sectors (i.e., industry groups) with n_i firms in each sector. Hence, $\sum_{i=1}^{n} n_i = N$. In our model, all stock returns refer to monthly gross returns. We consider a model where variations in stock returns are attributed to market, sector, and firm-specific factors. Thus, three stochastic components define the return on a stock. At time t, firm j in sector i has a gross return of

$$r_{i,j,t} = m_t + s_{i,t} + f_{i,j,t} \tag{1}$$

where m_t, $s_{i,t}$, and $f_{i,j,t}$ are the three uncorrelated components of return due to market, sector, and firm-specific movements, respectively. The market returns, m_t, are assumed to be normally distributed with mean θ and variance ν^2. Second, sector movements, $s_{i,t}$, are assumed to be normally distributed with mean zero and variance σ^2. And third, firm-specific movements, $f_{i,j,t}$, represent firm-specific shocks in the return and are also assumed to be normally distributed with mean zero and variance $\tau^2_{i,j}$. We can assume nonzero means for the market, sector, and firm-specific components that are not separately identified, and that do not impact the analysis contained in this paper. Stock returns for firms within a particular sector are highly correlated, while returns from different sectors are less correlated.

Strategy

By assumption, the benchmark index portfolio consists of n sectors with n_i firms within each sector. In a hypothetical perfect market with no transaction costs, the ideal way to design an index portfolio would be to own every asset in the benchmark portfolio. However, in practice, it is unnecessarily costly to purchase and hold *every* stock in the benchmark. Instead, it is more practical to hold a representative sample of the index. For simplicity, in our model, each sector will be represented in the portfolio by just one stock from that sector. Hence, we construct a portfolio of n securities, each representing a different sector. In addition to its better tracking ability, another benefit of holding stocks from various sectors is that the two nonsystematic risks (sector-specific and firm-specific risks) are reduced to an extent by diversification.

To replicate the benchmark return and minimize the TE, our strategy is to increase after-tax returns by using capital losses to balance capital gains. To this end, we introduce the x-percent rule in this model. The rule works as follows: Once the price of a stock position in the portfolio drops x percent from its purchase price, the portfolio manager sells that position and purchases another stock from the same sector to replace it. This capital loss is used to net against capital gains for the period. In other words, the x-percent rule is a tax-loss-generating rule.

Initially, the investor constructs a portfolio designed to closely track the index by holding one stock from each sector. So, the portfolio consists of a total of *n* stocks. While constructing the portfolio, the investor first chooses the proper stocks from each sector and sets the weights of these stocks so that he minimizes the TE and, second, sets a value for *x* that will minimize the *loss*, which is the sum of the negative amount of tax losses generated, transaction costs, and costs due to TE. Every month, the investor follows the changes in the prices of the stocks that he is holding. Once the price of a stock drops by *x* percent, he sells this stock and replaces it with a similar stock from the same sector. This is the tax (loss) generation process that the investor can use to minimize capital tax losses. If there are no transaction costs, the investor will make as many transactions as needed to minimize tax losses. However, there *are* usually transaction costs associated with each trade. Therefore, transaction costs are part of the cost incurred when implementing the *x*-percent rule. Furthermore, if trading every month severely disturbs the tracking ability, then the investor should consider this as part of the cost of generating tax losses. The investor's main objective in this model is to determine the size of *x* that minimizes the *loss function* to maximize the after-tax return of the index portfolio.

Definitions

This section formalizes the investor's problem and provides the definitions and notation needed for our model. Moreover, the objective function and its constraints are defined formally. The benchmark is a value-weighted index. The time *t* return on the benchmark is

$$r_{b,t} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \omega_{i,j,t-1} \, r_{i,j,t} \tag{2}$$

where $\omega_{i,j,t-1}$ is the weight set at time $t-1$ and $r_{i,j,t}$ is the return at time *t* on stock *j* of sector *i*. Time *t* return on the portfolio is

$$r_{p,t} = \sum_{i=1}^{n} \omega_{i,t-1} r_{J(i),t} \tag{3}$$

where $J(i)$ is the stock selected from sector *i*, $\omega_{i,t-1}$ is the time $t-1$ portfolio weight, and $r_{J(i),t}$ is the time *t* return on stock $J(i)$. Given that $J(i)$ is the stock selected from sector *i*, $\omega_{i,J(i),t-1}$ denotes the index weight on that stock $J(i)$. Also note that

$$\sum_{i=1}^{n} \sum_{j=1}^{n_i} \omega_{i,j,t-1} = 1 \tag{4}$$

and

$$\sum_{i=1}^{n} \omega_{i,t-1} = 1 \tag{5}$$

Let $\omega'_{i,t-1}$ be the weight of the sector *i* at time $t-1$ in the benchmark index. Then

$$\omega'_{i,t-1} = \sum_{j=1}^{n_i} \omega_{i,j,t-1} \tag{6}$$

with

$$\sum_{i=1}^{n} \omega'_{i,t-1} = 1 \tag{7}$$

We use the mean squared error as a measure of the TE in this study. The TE at time *t* is

$$TE_t = \left(\sum_{i=1}^{n} \sum_{j=1}^{n_i} \omega_{i,j,t-1} \, r_{i,j,t} - \sum_{i=1}^{n} \omega_{i,t-1} \, r_{J(i),t}\right)^2 \tag{8}$$

where $J(i)$ represents the stock chosen from sector i and $r_{J(i),t}$ is the return on this stock. Then *tracking error* for an investment horizon T is defined by

$$TE = \frac{1}{T}\sum_{t=1}^{T} e_t^2 \quad (9)$$

where T is the investment horizon and $e_t = r_{b,t} - r_{p,t}$.

We also need formal definitions for a tax loss, transaction cost, and the loss functions. Suppose that stock $J(i)$ is in the investor's portfolio. Also suppose that, at time t, stock $J(i)$'s price drops x percent relative to its purchase price. Then the investor sells this stock and repurchases another one from the same sector, while applying the x-percent rule at time t. The costs of trading the stock include taxation and transaction cost. We can define the tax loss and the transaction cost generated by this transaction. The *tax loss* for an investment horizon T is defined as

$$TL = \frac{1}{T}\sum_{t=1}^{T} \sum_{J(i)\in A} a\Delta V_{J(i),t} \quad (10)$$

where a denotes the tax rate applied to capital gain or loss from stocks, and $\Delta V_{J(i),t}$ is defined as the value of the stock $J(i)$ at the time the investor buys the stock less the value of the stock $J(i)$ at time t. Notice that in this paper the *tax loss* is defined as the positive dollar amount generated from applying the x-percent rule as this definition is more convenient to use in this study.

Transaction costs include the costs from two transactions, one from selling the old stock and another from buying the new one. The average *transaction cost* for an investment horizon T is defined as

$$TC = \frac{1}{T}\sum_{t=1}^{T} \sum_{J(i)\in A} 2(b + cV_{J(i),t}) \quad (11)$$

where b is the fixed cost, c is the cost proportional to the dollar volume traded, and $V_{J(i),t}$ is the dollar amount of stock $J(i)$ at time t. Next, the *loss function*, or abbreviated the *loss (L)*, is defined as

$$L = -TL + TC + g(TE) \quad (12)$$

where $g(\cdot)$ is monotonically increasing function for the TE. The function, $g(\cdot)$, can be interpreted as the *tracking error cost* and, in our case, is given by $g(TE) = d\sqrt{(TE)}$ where d is the coefficient of the tracking error which determines the level of TE cost. The investor is holding an index fund portfolio, so he cares about the tracking error of the portfolio. That is why TE enters the loss function as another cost. The number and volume of trade every month depend upon the size of x. Thus, the loss function L is a function of x. Ultimately, *the investor's objective is to find the value of x that will minimize the expected loss L.*

THE TRACKING ERROR PROBLEM

In the TE literature, there are many studies using rebalancing of the weights at the end of every period since they do not consider the transaction costs (Rudolf, et al., 1999; Ritter and Chopra, 1989). In this section, we also assume no transaction costs, but there is still a tax loss generating process. So, we rebalance the weights monthly.

Firms that belong to the same sector are similar in terms of return distributions. We represent the similarities within the sector by a simplifying assumption in this section. Without loss of generality, we assume that each stock in the same group has the same firm-specific shock variance. This will help the reader with analytical tractability.

There are a couple of significant problems we would like to investigate first. These problems are related to the selection of firms and the allocation of the investment in the fund. First, we will give the decomposition of the expected TE which will be referred to quite often in this section and in later sections.

Theorem 1. *Expected tracking error at time t can be decomposed into the following two terms:*

$$E[TE_t] = \sum_{i=1}^{n}(\omega'_{i,t-1} - \omega_{i,t-1})^2\sigma_i^2 + \sum_{i=1}^{n}[\sum_{j\neq J(i)}^{n_i} \omega_{i,j,t-1}^2 + (\omega_{i,t-1} - \omega_{i,J(i),t-1})^2]\tau_i^2$$

where the first term can be interpreted as the sector-specific risk component, and the second term is the asset-specific risk component.

Proof. The expected error is

$$E(e_t) = E(r_{b,t}) - E(r_{p,t}) = \sum_{i=1}^{n}[\omega'_{i,t-1} - \omega_{i,t-1})\theta] = 0$$

while the variance of the error is

$$\begin{aligned}
&Var(e_t) = Var\left(\sum_{i=1}^{n}\sum_{j=1}^{n_i}\omega_{i,j,t-1}\, r_{i,j,t} - \sum_{i=1}^{n}\omega_{i,t-1}\, r_{i,J(i),t}\right) \\
&= Var(\sum_{i=1}^{n}\sum_{j=1}^{n_i}\omega_{i,j,t-1}\, r_{i,j,t} + Var(\sum_{i=1}^{n}\omega_{i,t-1}\, r_{J(i),t-1}) - \\
&2Cov(\sum_{i=1}^{n}\sum_{j=1}^{n_i}\omega_{i,j,t-1}\, r_{i,j,t}\sum_{i=1}^{n}\omega_{i,t-1}\, r_{J(i),t}) \\
&= \left(v^2 + \sum_{i=1}^{n}\omega'^2_{i,t-1}\,\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\omega_{i,j,t-1}^2\,\tau_i^2\right) + \left(v^2 + \sum_{i=1}^{n}\omega_{i,t-1}^2\left(\sigma_i^2 + \tau_i^2\right)\right) \\
&-2(v^2 + \sum_{i=1}^{n}\omega'_{i,t-1}\,\omega_{i,t-1}\sigma_i^2 + \sum_{i=1}^{n}\omega_{i,t-1}\omega_{i,J(i),t-1}\,\tau_i^2) \\
&= \sum_{i=1}^{n}(\omega'_{i,t-1} - \omega_{i,t-1})^2\sigma_i^2 + \sum_{i=1}^{n}[\sum_{j=1,j\neq J(i)}^{n_i}\omega_{i,j,t-1}^2 + (\,\omega_{i,t-1} - \omega_{i,J(i),t-1})^2]\tau_i^2
\end{aligned}$$

where $\omega_{i,J(i),t-1}$ is the index weight on the stock *J* (*i*) selected from sector *i*. Then the expected TE at time *t* with *x* percent rule is

$$E[TE_t] = E(e_t^2) = E(e_t)^2 + Var(e_t)$$

$$= \sum_{i=1}^{n}(\omega'_{i,t-1} - \omega_{i,t-1})^2\sigma_i^2 + \sum_{i=1}^{n}[\sum_{j=1,j\neq J(i)}^{n_i}\omega_{i,j,t-1}^2 + (\,\omega_{i,t-1} - \omega_{i,J(i),t-1})^2]\tau_i^2$$

The first problem to be investigated is the selection of firms from the benchmark index. In other words, the investor shall figure out the answers to the following questions first. Which stock shall he hold initially from each sector? After a trade is triggered based on the *x*-percent rule at each period *t*, which stock from a particular sector should he buy? The answers to these questions follow from the following theorem. Let *J*(*i*) denote the stock selected from sector *i*.

Theorem 2. *The stock with the largest market value in the sector should be held to achieve the minimum expected tracking error at every period t.*

Proof. In the expected TE in Theorem 1, $\omega_{i,J(i)}$ denotes the security *J* (*i*)'s weight in the index. Obviously, for each of the sectors *i* = 1 . . . *n* only the $\sum_{j=1,j\neq J(i)}^{n_i}\omega_{i,j,t-1}^2 + (\omega_{i,t-1} - \omega_{i,J(i),t-1})^2$depends upon which stock we hold from each sector, that is *J(i)*. For each *i*, both $\sum_{j=1,j\neq J(i)}^{n_i}\omega_{i,j,t-1}^2$ and $(\omega_{i,t-1} - \omega_{i,J(i),t-1})^2$ decrease as the investor picks a larger stock in market value which implies that the index weight, $\omega_{i,J(i),t-1}$ is the larger one. So, the sum $\sum_{j=1,j\neq J(i)}^{n_i}\omega_{i,j,t-1}^2 + (\omega_{i,t-1} - \omega_{i,J(i),t-1})^2$ decreases too when the larger stocks are held. Hence, $\sum_{i=1}^{n}[\sum_{j=1,j\neq J(i)}^{n_i}\omega_{i,j,t-1}^2 + (\,\omega_{i,t-1} - \omega_{i,J(i),t-1})^2]\tau_i^2$ falls when all of the *n* firms held from the *n* sectors are larger in market value. Thus, the larger is the stock held in the portfolio, the smaller is the expected TE at time *t*.

There are two significant implications of this theorem: (1) Initially, the investor should construct the portfolio by holding the largest assets of the sectors in market value to have a minimum expected TE; (2) While trading with the *x*-percent rule at any period *t*, the investor should replace the asset sold with the largest one from its sector, after excluding the one to be replaced, to minimize the expected TE.

The next step is the allocation of the investment to the selected stocks. How should an investor allocate his wealth to the stocks to have a minimum expected TE? What weight given to each stock in the portfolio minimizes the expected TE? The variances of the risk factors, σ^2 and τ^2, are quite effective in determining the weights. Assuming no transaction costs, the investor would rebalance his portfolio each month to minimize the expected TE. Given that the selected stock from sector *i* is *J(i)* for i = 1 . . . *n*, the investment allocation in the portfolio will be as given in the next theorem.

Theorem 3. *For any period t, the portfolio weights that minimize the expected tracking error under the x-percent rule are jointly determined by the sector weights, the index weights on the stocks held in the portfolio, and the variances of the risk factors and are given by*

$$\omega_{i,t-1} = \frac{\sigma_i^2}{\sigma_i^2+\tau_i^2}\,\omega'_{i,t-1} + \frac{\tau_i^2}{\sigma_i^2+\tau_i^2}\left[\omega_{i,J(i),t-1} + inc_{i,t-1}\right]$$

where $inc_{i,t-1}$ *is called the incremental part and i* = 1 . . . *n.*

Proof. The investor will minimize the expected tracking error $E[TE_t]$ derived in Theorem 1 subject to the constraints

$$\sum_{i=1}^{n} \omega_{i,t-1} = 1$$

and $0 < \omega_{i,t-1} < 1$. The solution of this minimization problem is

$$\omega_{i,t-1} = H_i + \frac{(\sigma_i^2+\tau_i^2)^{-1}}{\sum_{l=1}^{n}(\sigma_l^2+\tau_l^2)^{-1}}\left(1 - \sum_{l=1}^{n} H_l\right)$$

where $H_i = \frac{\omega'_{i,t-1}\sigma_i^2 + \omega_{i,J(i),t-1}\tau_i^2}{\sigma_i^2+\tau_i^2}$. This solution can equivalently be given by

$$\omega_{i,t-1} = \frac{\sigma_i^2}{\sigma_i^2+\tau_i^2}\,\omega'_{i,t-1} + \frac{\tau_i^2}{\sigma_i^2+\tau_i^2}\left[\omega_{i,J(i),t-1} + inc_{i,t-1}\right]$$

where

$$inc_{i,t-1} = \frac{(\tau_i^2)^{-1}}{\sum_{l=1}^{n}(\sigma_i^2+\tau_i^2)^{-1}}\left(1 - \sum_{l=1}^{n} H_l\right)$$

The optimal weight on stock *i* is the weighted average of the sector weight, $\omega'_{i,t-1}$, and the firm's *incremental index weight*, $\omega_{i,J(i),t-1} + inc_{i,t-1}$, where $inc_{i,t-1}$ is the *incremental* part representing the firm's idiosyncratic characteristics in the weight. They are weighted with respect to $\frac{\sigma_i^2}{\sigma_i^2+\tau_i^2}$ and $\frac{\tau_i^2}{\sigma_i^2+\tau_i^2}$. It says that if the variance of the sector-specific factor is quite large relative to the variance of the firm-specific factor, then the optimal weight on stock *i* is going to be heavily determined by the sector weight; or if the variance of the firm-specific factor is quite large relative to the variance of the sector-specific fac tor, then the optimal weight on stock *i* is going to be heavily determined by the firm's incremental index weight. This intuition can also be seen from the TE decomposition. The first term of the decomposition which includes sector-specific variance is minimized when we set the weights to sector weights. The second term, which includes firm-specific variance, is minimized when we minimize the difference between the portfolio weight and index weight on the stock held in the portfolio.

The theorems above address some important questions in TE literature. Holding stocks from each of the different sectors or industry groups is one of the approaches used to construct a portfolio that mimics an index. If an investor is holding stocks from various sectors, the portfolio weights would be jointly determined by the variances of the risk factors, the sector weights, and the index weights on the stocks held in the portfolio. Also, if an investor is deciding which stock to hold from each of the sectors, the answer would be to hold the largest stocks in market value. However, we should note that these results all depend on the assumptions made at the beginning of this section. Different and more realistic assumptions might lead to a different selection and rebalancing rules.

SIMULATED ASSET RETURNS

This section illustrates the methodology together with its practical implications. We demonstrate how the model finds x. The optimal value for x is found numerically, as an algebraic solution of the loss function is not possible. We first take a sample investment universe (index) and construct a portfolio from this universe following the rules and conditions stated earlier. All assumptions, definitions, and strategies discussed in the methodology section are still applied in this part of the study. In addition, we assume that there are transaction costs and investors are earning taxable returns.

The targeted index consists of 500 stocks, assuming 20 sectors and 25 stocks per sector. The initial sector weights are determined by the following procedure: Initial prices of all stocks are normalized to one. The number of shares of all 500 stocks is determined randomly by a uniform distribution. Then index weights on all 500 stocks are computed. Note that the benchmark is a value-weighted index. The total initial investment on the index portfolio (V_0) is $250,000. Initially, the simulation of the tax-efficient index portfolio is based on a 12-month holding period. Then, a sensitivity analysis of changing the holding period is done.

The long-term average monthly gross return for the U.S. stock market from 1926 to 2000 is 0.0101, with a monthly variance of 0.0034. For the simulation, we allocate this variance to the three factors of the return and obtain the following variance structure: the market variance (v^2) is assumed to be 0.0012; the variance for each sector i is assumed to be the same $\sigma^2 = 0.0011$; and for each i and j, the variance of the firm-specific component is assumed to be the same $\tau^2 = 0.0011$. Since these factors are uncorrelated, their variances will add up to equal the aggregate return variance. Although our initial results and discussion will be based on these values, we will also explore the effects of changing these relative variances in the results section. The market factor mean is assumed to be 0.0101, while the sector-specific and firm-specific factor means are assumed to be zero.

Tax rates that apply to the capital gains and losses on assets are dependent upon the investor's holding period. Capital gains and losses on assets held one year or less are taxed at a maximum rate of 39.6%, while capital gains and losses on assets held longer than one year are taxed at a maximum rate of 28.0%. Because our modeling first examines and compares investments held between 12 and 36 years, we assume a tax rate of 28.0% (that is, $a = 0.280$ in the model). Then, as an extension, we consider how the results might change for a scenario of a one-year or less holding period, with the higher tax rate. These rates are taken from Dammon and Spatt (1996).

Both the fixed and variable transaction costs vary as dollar volume traded changes. These varying transaction costs are given in Bhardwaj and Brooks (1992). For this study, we assume trades range between $10,000 and $20,000 per month. For these amounts, the appropriate transaction cost for our model is $62+0.003\times V$, where $b = 62$, $c = 0.003$ and V is the dollar amount traded. We use six different loss functions to observe the changes in the optimal x value. Loss functions are as follows

$$L = -TL + TC + d\sqrt{(TE)} \tag{13}$$

for six distinct d values: (a) $d = 0$, (b) $d = V_0/5$, (c) $d = 2V_0/5$, (d) $d = 3V_0/5$, (e) $d = 4V_0/5$, (f) $d = V_0$. The value of coefficient d should be determined by the answer to the following question: "how important is tracking error for the investor?" If the tracking error is very important for the investor, then d will be higher and thus there will be a higher weight on *TE* cost. To see the results for different types of investors, outputs for six different weights on tracking error are computed and studied.

Finally, to complete the model, we define the loss rate and the tax loss rate. The *loss rate* is the ratio of the difference between total tax loss and total transaction costs in an investment period to the initial investment amount. The *optimal loss rate* is the value of the loss rate computed for the optimal x value. The *tax loss rate* is the ratio of total tax loss in an investment period to the initial investment amount.

We conduct 4,000 Monte Carlo simulations. Using the simulated asset returns, we solve numerically for the optimal x that minimize both loss functions. Moreover, optimal loss rates and tax loss rates associated with these optimal loss rates are also computed to see how well an investor can decrease tax losses. We take 200 grid points for x with 0.1 increments starting $x = 0.1$. Then we minimize the loss functions over x. To construct the optimal portfolio for minimizing tax losses, we answer the following questions: Which stocks should the investor hold initially? What is the portfolio weight on each stock? When trading based on the x-percent rule, which stock should be purchased to replace a sold stock?

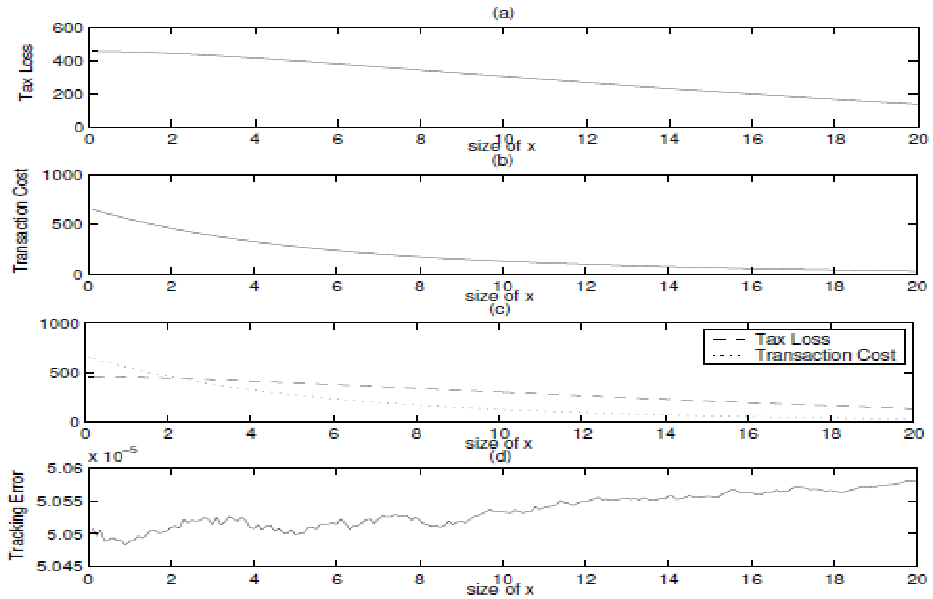
Since the expected return on the index is equal to the expected return on the portfolio, the investor will only consider the variance of the deviation in the optimization problem. Therefore, following the results derived earlier, to minimize the expected tracking error, (1) initially, the investor should construct a portfolio by purchasing one stock from each sector of the index of the company having the largest market capitalization in that sector; (2) portfolio weights are weighted averages described above; and (3) each time a stock is sold based on the x-percent rule during period t, it should be replaced by stock in the next largest company from the same sector. Now we are ready to examine the empirical results derived from the study using the simulated asset return data.

RESULTS

The results from the simulated asset returns described in the previous section are summarized in Figures 1–2 and Tables 1–5. (Note, because of the large size of the tables, they are in the Appendix.) The results presented in Figures 1–2 and Table 1 are derived under the standard case scenario, explained above. Then, Tables 2–5 report the sensitivity analysis results for various investment periods, stock volatilities, relative variances, and different tax rate assumptions. Results reported in Figures 1–2 are based on 4,000 independent replications. The minimum, mean, and maximum of the tracking error over x presented in the tables are also derived using 4,000 independent replications. To ensure accuracy in the computations of means and standard errors of optimal x values, loss rates, and tax loss rates, we split up 4,000 replications into 40 samples of 100 draws. Then the optimal x values, optimal loss rates, and tax-loss rates are computed using each of the 100 draws. Then means and standard errors are calculated using these 40 optimal values.

The panels in Figure 1 show the graphs of tax loss, transaction cost, and tracking error as a function of x, the percent loss that triggers a sale of an investment in the tracking portfolio. Both tax loss and transaction cost are decreasing functions of x. Furthermore, the tax loss curve is concave except where it converges to zero as x gets closer to 40 (not shown in the figure). The transaction cost curve is convex. The optimal difference between the loss and cost functions occurs at $x = 10.2$, which is the solution to the loss functions (see Figure 2 and Table 1A). Let's consider Figure 1C. In the region between tax loss and transaction cost for $x \in [0, 2.3]$, the loss function given in (a) is positive, so the investor is generating a negative return on his capital relative to the benchmark. In the region between tax loss and transaction cost for $x \in [2.3, 40]$, the loss function given in (a) is negative, so the investor can increase after-tax returns. For $x > 40$, the loss function becomes almost zero. As a result, the cut-off point to generate losses is $x = 2.3$ if we assume that TE is insensitive to sizes of x, that is equivalently $d = 0$.

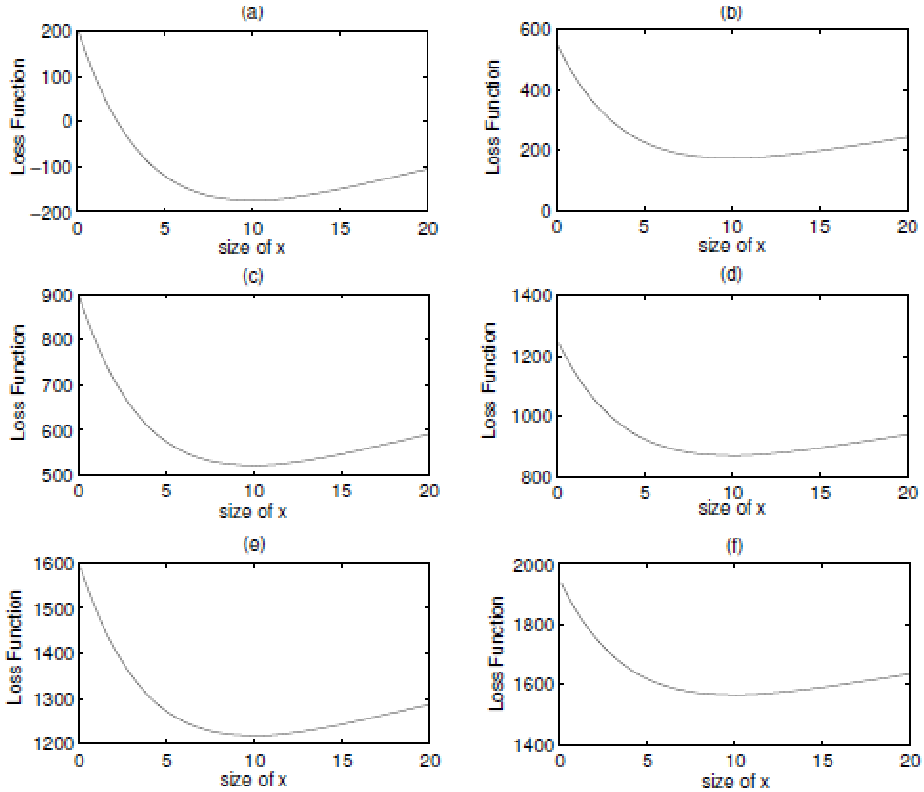
FIGURE 1
TAX-LOSS, TRANSACTION COST, AND TRACKING ERROR



As evident from Figure 1D, the TE is only slightly sensitive to changes in *x*. Let us see this more formally. In the simulations, the standard error of the TE for any size of *x* is approximately 3.3×10^{-7}. The minimum and maximum values of TE over *x* are respectively 5.048×10^{-5} and 5.058×10^{-5}, as reported in Table 1C. The distance between the minimum and the maximum is 10^{-7}, which is even smaller than one third of the standard error of the tracking error for any size of *x*. This implies that the variations in the tracking error are extremely small.

We can also observe the insensitivity of tracking error to changes in *x* also from loss functions exhibited in Figure 2 and outputs presented in Table 1A and Table 1B. From Figure 2A through Figure 2F, the weight on TE cost in the loss is increasing; the lowest possible weight ($d = 0$) in Figure 2A and the highest possible weight ($d = V_0$) in Figure 2F. In all six graphs, there is no change in the loss functions but their levels. So, if we add the TE term into the loss functions with different weight coefficients, only the levels of the functions are changing; the loss curves, mean and standard errors of the optimal *x* values and the optimal loss rates do *not* significantly change (see Table 1A and 1B). While we do not rule out the possibility that the magnitude of *x* influences the significance of the TE, our simulations suggest that variations in TE are miniscule and the TE does not influence the results. Consequently, the TE behaves like a constant in the loss function, as we minimize the loss with respect to *x*. Therefore, we can omit the TE from the loss function leading to the following result.

FIGURE 2
LOSS FUNCTION FOR SIX DIFFERENT COST OF TRACKING ERROR CASES: (A) $d = 0$, (B) $d = V_0/5$, (C) $d = 2V_0/5$, (D) $d = 3V_0/5$, (E) $d = 4V_0/5$, (F) $d = V_0$



By definition, minimizing the loss with respect to *x* is equivalent to maximizing the loss rate with respect to *x*. For this reason, we also report and discuss the optimal loss rates as well as the tax loss rates (associated with these optimal loss rates). However, one should notice that maximizing the loss rate does not imply that the tax loss rate is also maximized.

The six different loss functions presented in Figure 2 represent six different computations of loss, each of which corresponds to a different evaluation of TE cost. As we explained above, the level of TE cost does not affect the results but the level of the loss function. The *x* values that minimize the losses are almost identical for all six loss functions and are approximately 10.2 percent.

Table 1A shows means and standard errors of the optimal *x* values for each of the loss functions. The optimal size of *x* is approximately 10.2 regardless of which loss function is used (that is, regardless of the level of TE cost in the loss function); note that standard errors are around 0.1. Table 1B reports means and standard errors of the optimal loss rate and the tax loss rate associated with this optimal rate. The optimal loss rate is 0.00841 and the tax loss rate is 0.01453. Table 1C reports the minimum, mean and maximum values of the TE over *x*. Note that the standard error of the TE for any value of *x* is approximately 3.3×10^{-7}.

To understand the practical implications of these findings, suppose a fund manager has a client who wants to invest V_0 dollars in his tax-efficient index portfolio. What can the fund manager tell the client about this fund? He can say he is 95% percent confident that the optimal value of x is between 10.034 percent and 10.426 percent, and the optimal loss rate will be between 0.82 percent and 0.86 percent. The fund manager can tell the client how much he may lose and how much losses he will generate and how much of the tax loss he can recover, based on the *x*-percent rule. Suppose trading is based on $x = 10.23$ percent, the optimal value given for the case $d = 0$. Provided that the investor will liquidate the portfolio at the end of the investment period, the expected capital gain on his investment is $0.13 \times V_0$. The tax payment on this gain is $0.28 \times 0.13 \times V_0$. The amount of loss generated is $0.00841 \times V_0$, the net amount of tax loss recovered with x = 10.23 percent. The amount of tax loss generated is $0.01453 \times V_0$, so this is the gross amount of tax loss recovered when applying the 10.23-percent rule, i.e., transaction costs are not deducted from this amount yet.

Two ratios help quantify the performance of this trading strategy. First, the net rate of tax loss recovered, which is the ratio of the net amount of tax loss recovered to the tax amount paid, is $\frac{0.00841 \times V_0}{0.28 \times 0.13 \times V_0} = 0.231$. Second, the gross rate of tax loss recovered, which is equal to the ratio of the gross amount of tax loss recovered to the tax amount paid, is $0.01453 \times V_0 = 0.399$. So, if the client invests in this fund, he may recover 39.9% of the tax amount paid and ultimately recover 23.1% of his tax payments after deducting the transaction costs and minimizing the tracking error. These two rates are quite useful because sometimes (for example, when investment horizon or tax rate changes) both the net amount of tax loss recovered and tax amount paid simultaneously change and these two rates will tell us how much the investor is better off in terms of recovering the tax losses.

The Effect of Changing Investment Period

To explore the sensitivity of the results to the changes in investment period, volatility of the stock, relative variances (that is, the variances of market shock, sector shock, and firm-specific shock), and tax level, we examine results presented in Tables 2 through 5.

Table 2A summarizes the results of the changes in the investment period. The results indicate that as the investment period lengthens, the optimal x increases, meaning that long-term investors trade less frequently in this fund than short-term investors. The intuition behind this result is as follows: first, once the investor trades using the x-percent rule, he restarts the holding period for the x-percent rule. Short-term investors do not wait long to harvest larger losses because they do not have that much time to wait for large future loss opportunities to occur. Therefore, they trade less often to realize the best loss opportunities that occur in the short term. However, long-term investors wait longer and are reluctant to realize small current losses for fear of losing the opportunity to realize substantially larger losses in the future. Therefore, long-term investors should trade less often to realize better loss opportunities (today as well as in the future). Thus, the optimal x for a long-term investor is higher than for a short-term investor. This result is consistent with the discussions of Dammon and Spatt (1996).

Table 2B reports optimal loss rates and tax loss rates for different investment periods. As T rises, optimal loss rates (hence, the expected losses) and tax loss rates increase because when we wait longer, we will generate more losses. Yet, expected capital gains also increase in the long-term and, thus, expected tax payments are higher for the long-term. This raises the following question: Is the long-term investor better off than the short-term investor in terms of recovering tax losses? Mean gross returns on stocks for investment periods 12, 18, 24, 30, and 36 months are, respectively, 1.13, 1.20, 1.26, 1.35, and 1.44. Net rates of tax losses recovered for these periods are, respectively, 0.231, 0.190, 0.166, 0.135, and 0.116. Expected capital gains and, thus, expected tax payments grow faster than the expected losses. Hence, as the investment period gets longer, net rates of tax losses recovered fall whereas loss rates rise.

Table 2C reports the minimum, mean and maximum values of TE over x. TE is higher for longer investment periods since we do not rebalance the portfolio weights to minimize the tracking error at every period and, consequently, the return on the portfolio departs further from the targeted return, which is the return on the index as time passes.

The Effect of Changes in the Volatility of the Stock

Now, we explore the sensitivity of the results to the changes in stock volatility summarized in Table 3. One can expect that the size of the optimal x value for the x-percent rule is quite sensitive to market volatility. To assess this, we compare two different market variance assumptions to the base case, i.e., the variance of the long-term average gross return on U.S. capital markets of 0.00340. First, we test a smaller variance of 0.00085, which is one-fourth of the variance of the base case, and then we test a larger variance of 0.01360, which is four times the variance of the base case.

As the variance of the stock decreases, the likelihood of producing large losses goes down. In turn, the frequency of trading using the *x*-percent rule falls. Thus, investors trade with a lower *x*-percent rule when the stocks are less volatile. However, if the variance is larger, then the stocks produce larger losses within a specific investment period so that investors can realize higher loss opportunities in that period. So, since large future losses are more likely to occur with higher volatility, the optimal trading value, *x*, can be increased to realize higher losses. Hence, when the variance of the stock is higher, investors will have the opportunity to generate higher losses by trading with a higher *x* value and, thus, are willing to wait for better loss opportunities occurring in the future.

Table 3A provides the optimal trading values for *x* for the three values of the variance. For the higher stock volatility assumption, the optimal trading values are higher and more volatile. Furthermore, the higher variance leads to more trades for every level of *x*, allowing investors to realize higher losses by trading with higher optimal trading values, which will eventually increase the amount of loss generated. Hence the optimal loss rate will be bigger. Table 3B reports the effect of a change in the volatility of the stock on the optimal loss rates, confirming that optimal loss rates are bigger at higher stock variances. Table 3C presents the effect of changes in the volatility of a stock on the tracking error. Highly volatile stocks produce high tracking error, whereas less volatile stocks lower the tracking error.

The Effect of Changes in Relative Variances

Next, we explore whether changes in relative variances change the results by considering 10 different variance allocations. The results appear in Table 4. Our analysis includes extreme cases, such as $v^2 = 0.0002$, $\sigma^2 = 0.0002$, and $\tau^2 = 0.0030$, with its permutations, and we also include moderate cases, such as $v^2 = 0.0004$, $\sigma^2 = 0.0010$, $\tau^2 = 0.0020$, with its permutations. The standard case is the allocation of variance equally.

We can observe from results reported in Table 4A and Table 4B that the optimal values of *x*, the optimal loss rates, and the standard errors do not change significantly when there is a change in the relative variances. Hence, only the variance of the stock return itself matters; how the return variance is allocated does not matter in determining the optimal value for *x* and the optimal loss rate.

In contrast, the results shown in Table 4C indicate that the TE is highly influenced by the changes in relative variances. When the systematic risk factor (i.e., the market factor) has a high variance relative to other factor variances, the TE gets smaller because most of the variabilities in the index and portfolio offset each other. Oppositely, when the idiosyncratic firm or sector risk factors have high variances, the TE is higher because most of the variabilities in the index and portfolio do not offset each other. These can also be seen from the tracking error decomposition derived earlier. Although there are changes in the TE, optimal values for *x* and loss rates do not change because changes in the TE are level changes and optimal solutions are insensitive to these changes in TE, as discussed earlier.

The Effect of Different Tax Rates

The tax system in an advanced economy is designed to encourage long-term investments. Capital gains taxes are determined based on the holding period, and investors generally tend to hold their investments for at least one year to benefit from lower long-term capital gains tax rates. Following Dammon and Spatt (1996), we consider two tax rates, 28.0% and 39.6%, to explore the effect of different tax levels. As the tax rate is increased from 28.0% to 39.6%, the tax loss generated at every period increases, while the transaction cost remains the same. This leads to an increase in the level of the tax loss function. We find that with the higher tax rate, the optimal difference between the tax loss function and transaction cost function is realized at a lower value of *x*. Second, the loss and, thus, the optimal loss rate rises.

In Table 5A and Table 5B, we summarize the optimal *x* values and loss rates for both tax scenarios. There is a significant reduction in the optimal trading value, i.e., the size of *x* when the tax rate is increased from 28.0% to 39.6%. Trading with the rate of 28.0% results in an optimal *x* value of nearly 10.2, whereas trading with the rate of 39.6% leads to an optimal *x* value of approximately 7.8. Increasing the tax rate also increases the net rate of tax loss recovered from 23.1% to 29.4%. Thus, as the tax rate increases, the investor does not need to hold the assets longer to generate high tax losses.

We can interpret the effects of changing the tax rate as follows. The tax rate is the *price* of selling capital losses and the transaction cost is the *cost* of selling capital losses. As the tax rate rises, the revenue (i.e., amount of tax loss generated) from trading (i.e., selling the capital losses) increases while the cost remains the same. Consequently, trading is more attractive than before for an investor. Thus, the investor wants to trade more frequently. Note that changing the tax rate has no impact on the TE.

CONCLUSION

In this paper, we introduce a model to construct a tax-efficient indexed portfolio. For this purpose, we first study how to construct an optimal portfolio for minimizing tracking error. Second, we investigate an optimal trading strategy to maximize after-tax returns. The first part is an analytical study whereas the second part mostly gives quantitative results and is based on simulated asset returns.

The main contribution of this paper is to find an optimal trading strategy to improve the after-tax returns of an index portfolio. The strategy we discuss in the paper is to trade with an x-percent rule to generate tax losses that increase the after-tax returns. We observe that the strategy using the x-percent trading rule does not lead to any significant variations in the tracking error. Furthermore, the optimal trading strategy we find recovers a significant portion of the tax payments and therefore decreases the capital tax losses substantially.

There are possible extensions to this tax-efficient indexing research. For example, the benefit from identification of an optimal trading strategy with asymmetric capital gains taxes and the effects of changes in the number of stocks held from the sectors on the optimal trading strategy and loss rates could be examined. Also, distinct trading strategies (i.e., different optimal values for x) for each sector and each stock could be studied.

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APPENDIX: TABLE 1: THE STANDARD CASE

Standard Errors in Parentheses

TABLE 1A
OPTIMAL *X* VALUES

$d = 0$	$d = \frac{V_0}{5}$	$d = \frac{2V_0}{5}$	$d = \frac{3V_0}{5}$	$d = \frac{4V_0}{5}$	$d = V_0$
10.230	10.243	10.250	10.258	10.235	10.188
(0.100)	(0.103)	(0.108)	(0.109)	(0.106)	(0.119)

TABLE 1B
OPTIMAL LOSS RATES (10^{-3})

LOSS RATE	TAX LOSS RATE
8.41	14.53
(0.10)	(0.18)

TABLE 2C
TRACKING ERROR (10^{-5})

MINIMUM	MEAN	MAXIMUM
5.048	5.054	5.058

TABLE 2: THE EFFECT OF CHANGING INVESTMENT PERIOD ON

Standard Errors in Parentheses

TABLE 2A
THE OPTIMAL X VALUES

INVESTMENT PERIOD (T IN MONTHS)	$d = 0$	$d = \frac{V_0}{5}$	$d = \frac{2V_0}{5}$	$d = \frac{3V_0}{5}$	$d = \frac{4V_0}{5}$	$d = V_0$
12	10.230	10.243	10.250	10.258	10.235	10.188
	(0.100)	(0.103)	(0.108)	(0.109)	(0.106)	(0.119)
18	10.813	10.783	10.755	10.703	10.695	10.655
	(0.117)	(0.113)	(0.124)	(0.112)	(0.120)	(0.114)
24	11.540	11.505	11.483	11.485	11.475	11.493
	(0.118)	(0.118)	(0.124)	(0.122)	(0.125)	(0.130)
30	12.118	12.050	11.965	12.005	12.003	12.033
	(0.112)	(0.114)	(0.113)	(0.114)	(0.114)	(0.119)
36	12.195	12.115	12.060	11.983	11.940	11.878
	(0.132)	(0.132)	(0.144)	(0.147)	(0.150)	(0.148)

TABLE 2B
THE OPTIMAL LOSS RATES (10^{-2})

INVESTMENT PERIOD (T IN MONTHS)	LOSS RATE	TAX LOSS RATE
12	0.841	1.453
	(0.010)	(0.018)
18	1.064	1.791
	(0.012)	(0.020)
24	1.205	1.963
	(0.015)	(0.022)
30	1.323	2.109
	(0.013)	(0.023)
36	1.432	2.282
	(0.018)	(0.028)

TABLE 2C
THE TRACKING ERROR (10^{-5})

INVESTMENT PERIOD (T IN MONTHS)	MINIMUM	MEAN	MAXIMUM
12	5.048	5.054	5.058
18	5.078	5.082	5.087
24	5.131	5.138	5.144
30	5.182	5.190	5.198
36	5.225	5.233	5.241

TABLE 3: THE EFFECT OF CHANGING STOCK VOLATILITY ON

Standard Errors in Parentheses

TABLE 3A
THE OPTIMAL X VALUES

STOCK VARIANCE (10^{-2})	$d = 0$	$d = \frac{V_0}{5}$	$d = \frac{2V_0}{5}$	$d = \frac{3V_0}{5}$	$d = \frac{4V_0}{5}$	$d = V_0$
0.340	10.230	10.243	10.250	10.258	10.235	10.188
	(0.100)	(0.103)	(0.108)	(0.109)	(0.106)	(0.119)
0.085	7.468	7.438	7.435	7.420	7.383	7.323
	(0.070)	(0.071)	(0.073)	(0.072)	(0.069)	(0.071)
1.360	13.910	13.718	13.695	13.698	13.708	13.685
	(0.180)	(0.152)	(0.155)	(0.166)	(0.167)	(0.181)

TABLE 3B
THE OPTIMAL LOSS RATES (10^{-2})

STOCK VARIANCE (10^{-2})	**LOSS RATE**	**TAX LOSS RATE**
0.340	0.841	1.453
	(0.010)	(0.018)
0.085	0.089	0.237
	(0.002)	(0.005)
1.360	3.287	4.585
	(0.034)	(0.042)

TABLE 3C
THE TRACKING ERROR (10^{-5})

STOCK VARIANCE (10^{-2})	**MINIMUM**	**MEAN**	**MAXIMUM**
0.340	5.048	5.054	5.058
0.085	1.247	1.247	1.249
1.360	21.006	21.121	21.164

TABLE 4: THE EFFECT OF CHANGING RELATIVE VARIANCES ON

Standard Errors in Parentheses

TABLE 4A
THE OPTIMAL X VALUES

RELATIVE VARIANCES ν^2, σ^2, τ^2 (10^{-3})	$d = 0$	$d = \frac{V_0}{5}$	$d = \frac{2V_0}{5}$	$d = \frac{3V_0}{5}$	$d = \frac{4V_0}{5}$	$d = V_0$
1.2, 1.1, 1.1	10.230	10.243	10.250	10.258	10.235	10.188
	(0.100)	(0.103)	(0.108)	(0.109)	(0.106)	(0.119)
0.2, 0.2, 3.0	10.005	9.973	9.250	9.890	9.235	9.900
	(0.097)	(0.097)	(0.105)	(0.114)	(0.113)	(0.146)
0.2, 3.0, 0.2	9.905	9.908	9.250	9.928	9.235	9.928
	(0.076)	(0.076)	(0.077)	(0.080)	(0.081)	(0.081)
3.0, 0.2, 0.2	9.910	9.910	9.250	9.930	9.235	9.915
	(0.119)	(0.120)	(0.116)	(0.112)	(0.106)	(0.106)
0.4, 1.0, 2.0	9.913	9.913	9.250	9.803	9.235	9.768
	(0.072)	(0.074)	(0.075)	(0.087)	(0.108)	(0.110)
0.4, 2.0, 1.0	9.923	9.935	9.250	9.895	9.235	9.890
	(0.092)	(0.087)	(0.079)	(0.082)	(0.085)	(0.097)
1.0, 0.4, 2.0	10.210	10.170	10.250	10.025	10.235	9.930
	(0.098)	(0.093)	(0.097)	(0.103)	(0.115)	(0.121)
1.0, 2.0, 0.4	9.988	9.983	9.250	9.928	9.235	9.915
	(0.093)	(0.096)	(0.097)	(0.099)	(0.107)	(0.107)
2.0, 0.4, 1.0	10.043	10.023	10.250	10.008	10.235	10.020
	(0.105)	(0.106)	(0.108)	(0.110)	(0.113)	(0.111)
2.0, 1.0, 0.4	10.108	10.110	10.250	10.128	10.235	10.108
	(0.123)	(0.120)	(0.110)	(0.112)	(0.110)	(0.114)

TABLE 4B
THE EFFECT ON THE OPTIMAL LOSS RATES

RELATIVE VARIANCES ν^2, σ^2, τ^2 (10^{-3})	LOSS RATE (10^{-3})	TAX LOSS RATE (10^{-2})
1.2, 1.1, 1.1	8.41	1.453
	(0.10)	(0.018)
0.2, 0.2, 3.0	8.60	1.504
	(0.08)	(0.009)
0.2, 3.0, 0.2	8.51	1.499
	(0.06)	(0.011)
3.0, 0.2, 0.2	8.49	1.492
	(0.21)	(0.032)
0.4, 1.0, 2.0	8.45	1.485
	(0.06)	(0.012)
0.4, 2.0, 1.0	8.46	1.487
	(0.06)	(0.012)
1.0, 0.4, 2.0	8.41	1.455

	(0.09)	(0.016)
1.0, 2.0, 0.4	8.43	1.476
	(0.10)	(0.017)
2.0, 0.4, 1.0	8.54	1.489
	(0.16)	(0.027)
2.0, 1.0, 0.4	8.51	1.478
	(0.15)	(0.023)

TABLE 4C
THE EFFECT ON THE TRACKING ERROR (10^{-5})

RELATIVE VARIANCES ν^2, σ^2, τ^2 (10^{-3})	MINIMUM	MEAN	MAXIMUM
1.2, 1.1, 1.1	5.048	5.054	5.058
0.2, 0.2, 3.0	13.048	13.658	13.704
0.2, 3.0, 0.2	0.048	0.937	0.938
3.0, 0.2, 0.2	0.048	0.906	0.908
0.4, 1.0, 2.0	9.048	9.197	9.216
0.4, 2.0, 1.0	4.048	4.642	4.645
1.0, 0.4, 2.0	9.048	9.126	9.144
1.0, 2.0, 0.4	1.048	1.856	1.858
2.0, 0.4, 1.0	4.048	4.543	4.545
2.0, 1.0, 0.4	1.048	1.833	1.834

TABLE 5: THE EFFECT OF DIFFERENT TAX RATES ON

(Standard Errors in Parentheses)

TABLE 5A
THE OPTIMAL *X* VALUES

TAX RATE	$d = 0$	$d = \frac{V_0}{5}$	$d = \frac{2V_0}{5}$	$d = \frac{3V_0}{5}$	$d = \frac{4V_0}{5}$	$d = V_0$
a=0.280	10.230	10.243	10.250	10.258	10.235	10.188
	(0.100)	(0.103)	(0.108)	(0.109)	(0.106)	(0.119)
a=0.396	7.865	7.810	7.798	7.780	7.788	7.785
	(0.095)	(0.086)	(0.084)	(0.084)	(0.085)	(0.089)

TABLE 5B
THE OPTIMAL LOSS RATES (10^{-2})

TAX RATE	LOSS RATE	TAX LOSS RATE
a=0.280	0.841	1.453
	(0.010)	(0.018)
a=0.396	1.516	2.386
	-	-