Improvement of Reflection on Teaching Practice in a Training Course That Integrates the Lesson Study and Criteria of Didactical Suitability

Viviane Hummes
Universitat de Barcelona

Adriana Breda
Universitat de Barcelona

Vicenç Font
Universitat de Barcelona

María José Seckel
Universidad Católica de la Santísima Concepción

This article discusses a qualitative case study involving eight mathematics teachers in Brazil. The study aimed to analyze the effectiveness of a training course that combined Lesson Study and Criteria of Didactical Suitability in developing reflective competence in teachers. Lesson Study is a professional development strategy that encourages teachers to collaborate and reflect on their teaching practices, while Criteria of Didactical Suitability is a tool that guides teachers' reflection. The study aimed to evaluate the usefulness of Criteria of Didactical Suitability in enhancing reflection skills and to assess the participants' perception of its usefulness. The analysis revealed that participating teachers improved their reflection skills, specifically by utilizing Criteria of Didactical Suitability more effectively in the third phase of the course. However, the study has limitations, such as being conducted virtually and in a specific context. Overall, this study highlights the potential of combining Lesson Study and Criteria of Didactical Suitability in developing reflective competence in teachers. Further research can explore the efficacy of this approach in different contexts.

Keywords: teacher training, reflection on teaching practice, Lesson Study, Criteria of Didactical Suitability

INTRODUCTION

Many trends in teacher education, both initial and continuing, advocate for teachers to conduct research and reflect on their practice as a key component of their professional development. In this sense, from the Didactics of Mathematics, numerous proposals offer conceptual frameworks for the development of the reflective competence of teachers, such as the discipline of noticing (Mason, 2002), mathematical knowledge for teaching (Hill et al.., 2008), the concept study (Davis, 2008), the professional perspective

(Llinares, 2012), the Lesson Study (Huang, Takahashi & da Ponte, 2019) and the Criteria of Didactical Suitability (Godino, Batanero & Font, 2019).

Lesson Study (LS) (Isoda, Stephens, Ohara & Miyakawa, 2007) refers to a research activity carried out in the classroom (Burghes & Robinson, 2010; Da Ponte, Baptista, Vélez & Costa, 2012), which provides an opportunity for the development of reflection as the teaching activity is carried out. It is a teacher professional development strategy used to improve mathematics teaching and learning processes. On the other hand, the Onto-semiotic Approach to Mathematical Knowledge and Instruction (OSA) (Godino et al., 2019) provides us with the Criteria of Didactical Suitability (CDS) and its breakdown into components and indicators as a tool to structure the teacher's reflection. From this perspective, these criteria can guide the teaching and learning processes of mathematics and evaluate their implementation (Breda, Font & Pino-Fan, 2018; Breda, Pino-Fan & Font, 2017).

According to Breda, Hummes, Silva, and Sánchez (2021) and Hummes, Breda, and Font (2020), each of these approaches (LS and CDS) has advantages and limitations. One advantage is that the criteria teachers must consider at each stage of an LS cycle align with the CDS (Hummes, Breda & Seckel, 2019; Hummes, Breda, Seckel & Font, 2020). A plausible explanation is that the criteria proposed in each stage of the LS and in the CDS were developed by the mathematics education community who share a common understanding of what is vital to consider when carrying out the teaching processes and learning mathematics. However, the fact that not all the components proposed by the CDS are included in the LS stages (Hummes et al., 2020) may be because the guidelines and criteria of said stages are more general and, in certain sense, shallower, and this can be seen as a limitation.

Although in the courses that use the CDS (Font, Breda & Pino-Fan, 2017; Giacomone, Godino & Beltrán-Pellicer, 2018; Seckel & Font, 2020) the reflection of the teachers is organized through a broad guideline and detailed space is still needed where they can discuss and plan the implementation of a more in-depth didactic sequence, particularly in an environment of collaborative practices. In this regard, LS can greatly help improve the courses taught by CDS and provide a space for collective reflection (Hummes, 2022). The latter could be an extension of LS that creates a framework for the reflective process of the teachers.

On the other hand, it is essential to highlight that in Brazil, the context in which this research is carried out, this trend has also been followed on the reflection of teaching practice and collaborative work, which was formalized in 2019 through the guidelines of the *Base Nacional Comum Curricular (BNCC)*, which declares some principles for the development of skills and competencies of teachers in the school system. Among them, we highlight the declaration of the following principle: "to give more relevance to collaborative cultures, teamwork, new skills, reflective and investigative training." (Ministerio de Educación de Brasil, 2019, p. 18).

According to the background presented, this study sought to answer the question: how does a training course in which LS and CDS are articulated contribute to the development of reflection in Brazilian mathematics teachers? To answer this question, this study set out to achieve two objectives. In the first place, it was intended to characterize the level of development of reflection in mathematics teachers who participate in a training course that articulates these two theoretical approaches, and, secondly, it sought to analyze the evaluation of the participants regarding the usefulness of the CDS as a tool to guide reflection.

The second section of this paper presents the theoretical references used: LS and CDS. In the third section, the methodology used is explained, and the results are presented in the fourth section. And finally, it concludes with the conclusions, some reflections on the study and future perspectives (fifth section).

THEORETICAL FRAMEWORK

Lesson Study and Criteria of Didactical Suitability are the theoretical approaches presented in this section and a brief review of the related literature.

Lesson Study (LS)

The LS is an educational work method that is based on collaborative research and teaching practices among teachers. At the same time, these activities enhance student learning, and they also enhance educational practice, and drive the professional growth of teachers. It consists of the collaborative and detailed design of a research lesson, its implementation, and direct observation in the classroom, as well as a post-analysis (Fernández & Yoshida, 2004; Hart, Alston & Murata, 2011; Lewis, 2002; Murata & Takahashi, 2002; Wang-Iverson & Yoshida, 2005).

In an LS cycle, a group of teachers meets with a problem related to their students' learning, develop a research lesson (lesson or sequence of classes) to help students learn, and then study and discuss what they saw while implementing that lesson. Teachers have many opportunities to discuss student learning and how instruction affects it through numerous interactions.

International researchers report the existence of many models of LS cycles. An example of a cycle from LS (Murata, 2011) considers the following stages: a study of the curriculum and goal setting; lesson planning; implementation and observation of the lesson; collective critical reflection on the collected data; and design and implementation of a new lesson.

Several criteria need to be considered at each stage as a complete cycle unfolds from LS (Hurd & Lewis, 2011; Lim-Ratnam, 2013). On the stage *curriculum and goals,* the consultation should not be limited only to the curriculum; somewhat, it should be expanded to include a review of various teaching materials, the opinions of knowledgeable professionals in the field, as well as scientific studies that focus on teaching the topic of the lesson. In addition, it is recommended that you record this entire discussion and specify what concepts students should have previously learned concerning the topic being studied and what concepts they will learn in class. These preliminary studies serve as the basis for developing the learning objectives presented in the next stage (*lesson planning*).

The next step in the lesson planning stage, is to develop the specific objectives of the class, partly by researching the materials to be used and partly by developing the assessment methodology and approach. It should be clear at this stage how the planned lesson will enable students to achieve the overall learning objectives set and should anticipate student reactions and questions. At this stage, decisions are also made about data collection for the research to be carried out by the teaching group. Changes may be made in some aspects by the teacher conducting the lesson due to unforeseen circumstances or because they facilitate student acquisition of knowledge more than the original group plan.

During the stage of *implementation and observation,* one teacher instructs the class while the others observe and record the procedure. This teacher must accept that others can attend his class. Students actively participate in each stage of problem-solving, from understanding the problem to formulating strategies and analyzing the solution. It is a stage where much weight is given to problem-solving techniques. The teacher leading the class should encourage this stage by carefully guiding students to share their knowledge, analyze, compare, and contrast their ideas, considering factors such as effectiveness, generalizability, and similarity to what has already been learned.

In the stage of *critical reflection,* after the lesson has been delivered, the group that designed the lesson and other invited professionals who have observed its implementation meet to discuss how it has affected student learning. Each observer at that moment expresses his impressions about the knowledge production of the students.

After reflection, the group of teachers may adjust for a later lesson on the same topic. In this sense, the class planning must be changed with actions focused on improving what has been considered important. This marks the beginning of a new cycle that includes redesign and reimplementation and is situated at a more mature stage.

Criteria of Didactical Suitability (CDS)

The CDS proposed in the OSA is intended to partially respond to the following question: what criteria should be used to plan a sequence of activities that allow evaluating and developing students' mathematical competence, and what changes should be made in their redesign to improve the development of this competence?

In the OSA, the following CDS are considered (Godino et al., 2019): *epistemic suitability*, to assess whether the mathematics taught is "good mathematics"; *cognitive suitability*, to evaluate before starting the teaching process, if what you want to teach is at a reasonable distance from what the students know, and after the process, if the learning acquired is close to what was intended to be taught; *interactional suitability*, to evaluate if the interactions resolve the doubts and difficulties of the students; *mediational suitability*, to evaluate the adequacy of the material and temporary resources used in the teaching process; *affective suitability*, to assess the involvement (interests and motivations) of students during the teaching process; *ecological suitability*, to evaluate the adequacy of the teaching process to the educational project of the educational center, the curricular guidelines and the conditions of the social and professional environment.

The operation of the CDS requires the definition of a set of observable components and indicators (see Table 1), which allows evaluation the degree of suitability of each of these criteria. Breda et al. (2017) establish a system of indicators that serves as a guide for the analysis and evaluation of Didactical Suitability, which is intended for a teaching process at any educational stage and explains how these criteria and their respective components and indicators were generated (Breda et al., 2017; 2018).

TABLE 1
CRITERIA AND COMPONENTS OF DIDACTICAL SUITABILITY

Criteria of Didactical Suitability	Component
epistemic	(IE1) Errors; (IE2) Ambiguities; (IE3) Richness of processes; (IE4) Math Object Representation.
cognitive	(IC1) Previous knowledge; (IC2) Curricular adaptation to individual differences; (IC3) Learning; (CI4) High cognitive demand.
interactional	(II1) Teacher-students interaction; (II2) Interaction between students; (II3) Autonomy; (II4) Formative evaluation.
mediational	(IM1) Material resources; (IM2) Number of students, schedule and classroom conditions; (IM3) Time.
affective	(IA1) Interests and needs; (IA2) Attitudes; (IA3) Emotions.
ecological	(IEC1) Adaptation to the curriculum; (IEC2) Intra and interdisciplinary connections; (IEC3) Socio-labor utility; (IEC4) Didactic innovation.

The CDS must be understood as correction norms emanating from the argumentative discourse of the educational community when it is aimed at reaching a consensus or what can be considered the best (Godino et al., 2019). From this perspective, mathematics education can offer us provisional principles established from the consensus of the interested community, which can serve to guide, evaluate and assess mathematics teaching and learning processes. As explained in Breda et al. (2018), the current trends in the teaching of mathematics are a first way of observing the consensus in the mathematics education community since they can be considered as regularities found in the discourses that deal with the improvement of teaching processes and mathematics learning – by example, presentation of contextualized mathematics; give importance to the teaching of mathematical processes; active (constructivist) teaching and learning; principle of equity in compulsory mathematics education and incorporation of new information and communication technologies (Guzmán, 2007). These new currents crystallized in the curricula of different countries in the form of guidelines and principles that indicate, a priori, how to have a quality mathematics education (NCTM, 2000). In addition, they have had an impact on official guidance in different countries. In Brazil (context of this research), for example, the curricular guidelines of the *Base Nacional Comum Curricular* (Ministério da Educação do Brasil, 2018) propose the use of some of these trends for the teaching of mathematics in basic education, such as mathematical modeling, technological means, problem-solving, etc. On the other hand, in mathematics education, knowledge and results have been generated that enjoy a broad consensus within this community (Bikner-Ahsbahs & Prediger, 2010).

METHODOLOGY

This is a qualitative case study research (da Ponte, 2006) that sought to analyze, on the one hand, how a training course that articulates the LS and the CDS develops the reflection of a group of mathematics teachers and, on the other hand, to analyze the participant's assessment of the usefulness of the CDS as a tool to guide reflection.

Context and Participants

The course was implemented as a university extension (training course for practicing teachers) in a public institution of higher education in southern Brazil. This was dictated from March to July 2020 and had 15 weekly meetings of two hours and a half. It was planned to be face-to-face. However, after the first session, the Covid-19 pandemic was declared in Brazil. With the suspension ofin-person activities, it was restructured to be implemented virtually and synchronously through the digital platform Skype.

The team of research professors that designed and carried out the course comprised the first three authors of this work. In addition, the first author, from now on called the research professor, was the one who gave the course sessions to the participating teachers.

Eight mathematics teachers – P1, P2, P3, P4, P5, P6, P7, P8 – who practice their profession in schools in southern Brazil with students (11 to 18 years old) participated in the course voluntarily and with their informed consent. At the time of the course, all the participants had a mathematics teacher's degree and teaching experience of between three and fifteen years. Three of them also had a master's degree in mathematics education.

The course was structured to develop in three phases. In the first, two complete cycles of LS were developed, considering all its stages, that is a) choice of topic - Pythagorean Theorem, among other aspects, as it is one of the curricular contents that participating teachers could explain in their classes; b) study of the topic in the curriculum and establishment of learning goals; c) planning of the class; d) execution and observation of the class and e) reflection on the implemented class. It is important to note that the two cycles from LS were developed with the participating teachers divided into two groups: one of them to teach the Pythagorean Theorem for students aged 14-15, as a new subject to be studied (developed by participants P1, P2, P3, and P4) and the other to use the Pythagorean Theorem with students aged 17-18 (developed by the participants P5, P6, P7, and P8). In the second phase of the training course, the participants were taught the CDS as a tool to guide teacher reflection. Finally, in the third phase, the participants made a new reflection, analysis and redesign of the class implemented using the CDS. The following table (Table 2) summarizes the phases and stages of the developed course.

TABLE 2
PHASES AND STAGES OF THE COURSE

Phases of the course		Stage 1	Stage 2	stage 3	stage 4
1st phase	**cycle development from LS**	study of the curriculum and definition of goals	lesson planning	implementation and observation of the class	class reflection
2nd phase	**CDS teaching**	implicit CDS in cycles from LS	epistemic CDS	cognitive CDS	interactional, mediational, affective and ecological CDS
3rd phase	**use of the CDS as a tool to guide reflection**	new reflection with the CDS of the class implemented in the LS	redesign with the CDS of the class implemented in the LS	questionnaire about the course	evaluation and closing of the course

Data Collection

The data collection was carried out through two techniques: video recordings of all the virtual meetings and classes implemented in the LS cycles, and a questionnaire with open questions to evaluate the course. Said questionnaire was designed considering the following components: 1) evaluation of distance modality; 2) assessment of the LS; 3) valuation of the CDS; and 4) self-perception of the teaching skills developed, and their content was validated through expert judgment.

Data Analysis Procedure

The content analysis technique (Bardin, 1991) was applied for data analysis. In the first place, the analysis on the reflection of the group of teachers (obtained by video recordings of the course sessions) was carried out using, as in previous categories, the components and indicators of the CDS presented in Table 1 (theoretical framework session). For this analysis, the explicit use of the criteria, components, and indicators of the CDS was observed, analyzing the discourse of the participants in the sessions that correspond to the third phase of the course, where the participants collaboratively made the analysis, reflection, and redesign of the Pythagorean Theorem class for students aged 14-15. For this reason, assuming that a team member's reflection is discussed and agreed upon among all the members, the level of competence is analyzed considering the work team. Likewise, similar to what was done in Breda (2020), a discrete assessment scale (from 0 to 4) was established for each component of the six CDS, which was used to assess each unit of analysis after categorizing the speech of the participants. Table 3 explains the characteristics of each level of the scale.

TABLE 3
ASSESSMENT SCALE FOR THE EXPLICIT AND CORRECT USE OF EACH COMPONENT OF CDS

Assessment level for the explicit and correct use of each component of the CDS	Characteristics
0	In the reflection of the teachers there are no extracts of the speech that show the explicit and correct use of any indicator of the CDS component that is being analyzed.
1	The reflection of the teaching staff sporadically presents extracts of the discourse that can be considered evidence of the explicit and correct use of some indicator of the CDS component that is being analyzed.
2	In the reflection of the teachers, there are extracts of the speech that show the explicit and correct use of some indicator of the CDS component that is being analyzed.
3	In the reflection of the teachers, there are excerpts from the speech that show the explicit and correct use of most of the indicators of the CDS component that is being analyzed.
4	In the reflection of the teachers, there are very detailed and coherent extracts of the discourse that show the explicit and correct use of most of the indicators of the CDS component that is being analyzed.

Specifically, for the analysis of these results, the following steps were considered: 1) categorization of the discourse of the participants based on the CDS; 2) assessment of the discourse applying the data triangulation technique, that is, the first three authors analyzed separately and then agreed on their assessments by applying the assessment scale (Table 3), and 3) comparison of the assessments attributed to each component of the CDS in the first and third phase (results of the first phase are available in summary form in Hummes, Breda & Font, 2022, and expanded in Hummes, 2022).

On the other hand, regarding the data collected with the questionnaire technique, the participants' speech was analyzed, and categories related to the four components of this instrument were raised.

RESULTS

Development of Reflection in Mathematics Teachers in a Course That Articulates LS and CDS

Next, Table 4 shows the results of the level of development of reflection in the teachers participating in the study. In particular, the general results are shown, comparing the levels obtained in this study's first and third phases.

TABLE 4
COMPARATIVE TABLE OF THE LEVEL OF USE OF CDS COMPONENTS IN THE 1ST AND 3RD PHASES OF THE COURSE

CDS	Components	level of use		Difference
		1st phase	3rd phase	
Epistemic	Errors (IE1)	1	3	**+2**
	Ambiguities (IE2)	0	2	**+2**
	Process richness (IE3)	3	4	**+1**
	Math Object Representation (IE4)	3	3	**0**
Cognitive	Previous knowledge (IC1)	4	4	**0**
	Curricular adaptation to individual differences (IC2)	0	2	**+2**
	Learning (IC3)	2	3	**+1**
	High cognitive demand (IC4)	1	1	**0**
Interactional	Teacher-students interaction (II1)	4	4	**0**
	Interaction between students (II2)	0	4	**+4**
	Autonomy (II3)	1	3	**+2**
	Formative evaluation (II4)	2	4	**+2**
Mediational	Material resources (IM1)	3	4	**+1**
	Number of students, schedule and classroom conditions (IM2)	2	4	**+2**
	Time (IM3)	3	4	**+1**
Affective	Interests and needs (IA1)	2	4	**+2**
	Attitudes (IA2)	1	3	**+2**
	Emotions (IA3)	1	4	**+3**
Ecological	Curricular adaptation (IEC1)	1	4	**+3**
	Intra and interdisciplinary connections (IEC2)	4	4	**0**
	Social and labor utility (IEC3)	1	4	**+3**

Following the previous table, concerning the epistemic CDS, the results indicate that the errors (IE1) and ambiguities (IE2) components had a considerable improvement if we compare the reflections analyzed in the first and third phases of the course and that the richness of processes (IE3) and the math object representation (IE4), were also widely contemplated in the reflection that arose in the third phase.

For example, in the case of the IE2 component, it is observed that in the first phase, no units of analysis were presented that could be related to this component. However, in the third phase of the course, the development of reflections with this component is observed. The following excerpt is evidence of this:

> *P1: The way in which the Pythagorean Theorem was stated can generate an ambiguous understanding. [...] The angle less than 90º was presented as a "bit", and I think that could cause an ambiguous understanding. [...] When he spoke of a triangle that is not a rectangle, he gave an example, but not a geometric example. In that case it could also lead to ambiguous compression, there's no telling what this triangle might look like for example. [...] Another ambiguity is when he spoke of Pythagorean Theorem and did not explain what a theorem is. The students also have to understand, for example, that a theorem is a proposition that can be proved through a logical process. Therefore, there is also a need to present a demo.*

Regarding the cognitive CDS, Table 4 shows that, concerning the IC1, the teachers continued to take this component into account in the reflection of the third phase, which allowed them to design tasks that are at a shorter distance within the development zone close to the students. Regarding the IC2, we have gone from not thinking about the need to make curricular adaptations, to considering diversity, to being aware that this aspect was not present in its first reflection but that it must be considered when reflecting on the design and redesign of a lesson. The following transcript is proof of this:

> *P8: We noticed that the teacher always came back to help - I saw that there was a student who joined the call later and had not given part of the explanation - P4 picked up and did the whole explanation again. I think that, in this sense of curricular adaptations, what caught my attention the most was, in relation to having used different resources to make the explanation, that video was used, images were used, exercises were used.*

Regarding the IC3, in the third phase of the course, the teachers elaborated a redesign of the class with some evaluation instruments that allow demonstrating the learning of the students, which led to an increase of one unit in the level of reflection of this component regarding reflection in the first phase. About IC4, no improvement is noted in the reflection of this component beyond its indirect improvement derived from IC1 (zone of proximal development) and the increase in the "richness of processes" component of the epistemic CDS.

Regarding the interactional CDS, Table 4 shows that, in the third phase of the course, the participants obtained a level of explicit use of component II1 equal to 4 (it remained identical to that of the first phase); component II2 equal to 4 (increased four units); component II3 equal to 3 (increased two units compared to phase 1); component II4 equal to 4 (increased two units).

In the third phase of the course, regarding component II4, P6 highlighted that he considered that there was an interactive process between the teacher and the students that allowed a formative evaluation to be made (for example, delivery of photographs with material prepared by the students, activities and questions to be answered in the YouTube chat, where the class was held), as evidenced by the following extract:

> *Q6: An example of the observation that the teacher made about the cognitive progress of the students was when he asked them to send the images of their constructions. [...] The same is also observed when he paused to answer the questions that arose in the chat. [...] The applied questionnaire also made it possible to observe the progress of the students.*

Regarding the mediational CDS, Table 4 indicates that there was a significant improvement in the development of reflection in the third phase of the course, with the explicit CDS, since all the components increased their level of mediational suitability. The IM1 and IM3 components went from 3 to 4 and the IM2 component went from 2 to 4. The following comment from P3 is evidence regarding the IM2 component:

> *P3: I believe that there were a good number of students and that they were able to participate well. Regarding the schedule, which they had previously agreed with P4, it helped a lot so that as many students as possible participated. Regarding the conditions of the class, being online, the issue of the internet connection was not a problem and I consider it very adequate as well.*

Regarding the affective CDS, Table 4 highlights that there was a significant improvement in the development of reflection in the third phase of the course compared to the first phase, since all the components increased their level. The IA1 component went from 2 to 4, the IA2 from 1 to 3 and the IA3 from 1 to 4. Regarding the indicator "presents a selection of interesting tasks for students" of the interests and needs component (IA1), P5 commented the following:

> *P5: One positive thing was the use of caricatures to illustrate the life of Pythagoras. [...] Another thing that he used, that I considered positive, that caught the interest of the students, was that P4, when he went to assemble the triangle with the squares on each side to verify the Pythagorean Theorem geometrically, mentioned a puzzle and I found it interesting also, I liked that word, because it refers to a game, mainly a children's game, of the age of the students we were watching. [...] One thing that I did not find so positive was the duration of the class, which makes it difficult for students to participate. [...] The duration of the class was a bit in the way. [...] At first everyone responded and, later, participation decreased, which I think makes interest a bit difficult and interferes with understanding.*

Regarding the ecological CDS, Table 4 shows that, compared to the first phase of the course, there was a significant improvement in the development of reflection in the third phase, since all the components increased their level. The IEC1 and IEC3 components went from 1 to 4 and the IEC2 and IEC4 components maintained the same level at 4. Regarding this component, P4, in the third phase of the course, presented the following reflection:

> *Q4: The class was on YouTube, due to social distancing, and nothing had ever been done remotely at my school. [...] We made sure the class was on YouTube because it was a platform the students were already familiar with. We also use video, which is something I particularly can't use in my classroom, because I don't have a projector. [...] Another thing that we used as an innovation was the Google Form. [...] Also, WhatsApp because the students sent me videos and other things were arranged through WhatsApp. I sent the video link on WhatsApp, they sent me the photos on WhatsApp, the class time that would be possible was all there as well. So, we had some didactic innovation, in a certain way, and that, perhaps, in a classroom, could not have been so technological.*

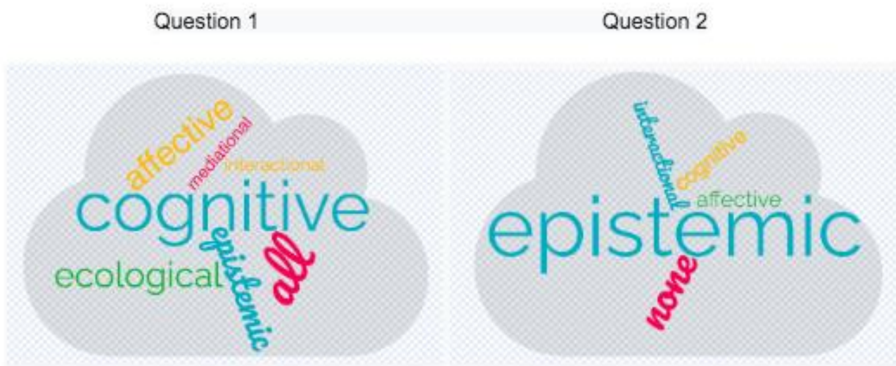
Evaluation of the Participants on the CDS as a Tool to Guide Reflection

Next, the results obtained regarding evaluating the participants on the CDS as a tool to guide reflection are presented. Likewise, we will comment on some relationships between this and the assessment of the distance modality, the assessment of the LS and the self-perception of the teaching skills developed.

To analyze the participants' assessment of the CDS, three open questions were considered: 1) Which of the CDS do you consider to be the most important for analyzing and assessing a mathematics class? Explain the reasons. 2) Among the six CDS studied, are there any or some that caused you difficulty understanding? 3) Do you consider that the CDS, components, and indicators studied cover the entire process of teaching and learning mathematics? If so, explain why. If not, which criterion, component, or indicator would you add or remove?

In this respect, Figure 1 shows, through a cloud of concepts, the result obtained in questions 1 and 2.

FIGURE 1
CLOUD OF CONCEPTS REGARDING QUESTIONS 1 AND 2



Question 1 shows that the concept with the most mentions in the participants' discourse was cognitive (4 out of 8), while the least mentioned were interactional and mediational (1 out of 8). Likewise, some participants (3 out of 8) indicated that all the CDS are important, and the others CDS (epistemic, affective, and ecological received two (2 out of 8) mentions each. Some evidence regarding the justification of the importance of cognitive CDS is presented below:

> *P5: The cognitive, because it is our role as a teacher to do what is necessary for the student to learn.*

> *P1: The cognitive allows to reflect on the previous knowledge of the students and makes the teacher think about resources that improve learning according to the level of the class, including all students.*

Regarding question 2, it was identified that the epistemic CDS was the most cited (5 out of 8) when defining the most difficult CDS to understand. Some evidence regarding the justification of these statements is shown below:

> *P1: The epistemic left me a little doubtful at first, but I managed to understand after doing the analysis and redesign of the class. Doubts arose due to the degree of complexity of this criterion. Differentiate the didactic error from the mathematical error, for example.*

> *P4: I think that the epistemic involves many concepts that I still need to learn more.*

> *P7: The epistemic seemed to me the most complex. I will study it again to seek to understand it better.*

> *P4: The epistemic, in the first part of the course but, throughout the discussions with the teacher and colleagues, it was clarified.*

In addition, one teacher also mentioned that he found the cognitive CDS difficult, and another teacher mentioned that the affective and interactional CDS have points that are very similar, which confused him a bit. Finally, two teachers said that they were able to gain a complete understanding of all of them. At the same time, the data shows that no participant assigned a degree of difficulty to the mediational and ecological CDS.

On the other hand, when asked if the CDS contemplates all the necessary criteria to promote a good process of teaching and learning mathematics (question 3), all the teachers stated that the CDS contemplates everything. Below are some pieces of evidence that exemplify the foundations provided by the participants:

> *P8: Yes, I believe that the criteria, together with their components and indicators, cover all the necessary principles for the teaching and learning of mathematics.*

> *P6: I think so. I even tried to think of something that wasn't covered, but couldn't find it.*

Based on the above, we will present some relationships between the assessment of the CDS as a tool for reflection and the other dimensions analyzed through the application of the questionnaire.

In relation to the evaluation of the distance modality, with respect to the appreciation of the participants about the CDS as a tool to guide reflection, it indicates that the teachers considered the course in this modality effective for the development and execution of the requested activities. However, some teachers emphasized that the course should have more hours because some sessions, especially those of the CDS teaching phase, were very dense and also required a lot of extra time. Finally, as suggestions for improvements, the teachers highlighted the extension of the course time, with shorter sessions and more weekly frequency.

Regarding the evaluation of LS, participants highlighted both positive and negative aspects. Among the positive aspects, the following stand out: collective reflection (during the design, observation, and redesign of the class), improving the teaching and learning processes of mathematics, and the understanding of the CDS through LS. On the other hand, concerning the negative aspects, the idea of time available considering the reality of the work context stands out. In this sense, we observe a relationship between the positive aspects they attribute to LS and the assessment presented above on the CDS, above all, because of the value they attribute to LS when it comes to an understanding the CDS. Some evidence of this is shown below:

> *P5: I learned a lot with the LS and the CDS. I think that from now on my classes will not be the same. I must consider many things that I probably knew, but had not realized that this was the case.*

> *P1: The LS made it possible to understand the CDS, especially the epistemic and cognitive, since I was able to perceive several processes that improved considerably in reflection with this tool.*

On the other hand, regarding the negative aspect, we observe a relationship with the assessment that the participants have given to the CDS in question 2 (see Figure 1), since in their arguments, they also refer to the time factor as a necessity to better understand the CDS, in particular, the epistemic one. The following is evidence that exemplifies the discourse of the participants in this line:

> *P1: I consider the time necessary for reflection to be a negative aspect of LS. I really liked the theory, but I realized that, in the institutions where I work, there would not be enough time to do collaborative work.*

Regarding the self-perception of the teaching skills developed during and through the course, the teachers highlighted, mainly, the development of mathematical knowledge through the epistemic CDS, which is observed in the following evidence:

P1: We improve meanings that are inserted in the Pythagorean Theorem such as the geometric, arithmetic-algebraic meaning and the need for validation of the Pythagorean Theorem with its demonstration.

P2: It was very relevant to understand the demonstration and, mainly, to differentiate a verification from the demonstration.

P6: We take an example about the relation between the areas and the relation of measure of the sides of the triangle. I had never before realized the different meanings, and that, normally in classes, we rarely work with exercises that involve the relationship between the areas.

With this, we observe a relationship with the participants' assessment of the CDS, because although they consider epistemic CDS to be the most complex to understand, at the same time it is the dimension that stands out when it comes to assessing the teaching competences developed.

On the other hand, they highlight the development of a broader knowledge to design, observe and redesign their classes using the CDS, which can be seen in the following evidence:

P5: I learned to take a closer look at the student and his needs (in all aspects). I believe that it is customary to think only from the cognitive point of view, that the students learn only that content. With the course I learned to look at the other criteria and I am already implementing them in my plans.

P1: The course provided us with rich reflections on the teaching and learning process and I could see the need to use each criterion to verify that the class was good, I was able to collaborate with the proposal/strategy thinking about the classes we had, even using my analysis to improve the construction of the Pythagorean Theorem test contextualized to the class we planned. In addition to participating in the whole redesign, reaching consensus and giving suggestions.

P3: These criteria made me see the class in a much broader and more professional way. Knowing the CDS I was able to analyze the class with a different look.

P2: The course and the CDS helped me reflect and think about problems that I don't know how to face and rethink others that I hadn't noticed.

P6: Before the course I was already very demanding with myself, but after the course I became even more demanding. Paying attention to details, for example in the diversification of exercises, that is, exercises that work on different concepts on the same topic, I have done this before, but because of my belief, now I have more clarity about the importance of this. I try to avoid ambiguous questions and pay more attention to my speech.

CONCLUSIONS

The objective of this study, on the one hand, was to characterize the level of development of reflection in mathematics teachers who participate in a training course that integrates LS and CDS. In addition, we sought to analyze the participant's assessment of the usefulness of the CDS as a tool to guide their reflection.

The analysis indicates that the teachers, when participating in the training course, showed an improvement in the development of reflection on the teaching of the Pythagorean Theorem, evidencing a better and more profound level of use of the CDS in the third phase. On the other hand, the levels of use of the CDS reached by the participants in the third phase of the course indicate that they can be taught and also learned, a result that coincides with other experiences carried out in the context of the initial and continuous training of the teachers (Esqué & Breda, 2021; Godino, Giacomone, Font & Pino-Fan, 2018; Morales-Maure, Durán-González, Pérez-Maya & Bustamante, 2019; Seckel & Font, 2020).

This result also coincides with research that highlights that teachers need tools to direct their attention to aspects related to the teaching and learning processes and that these tools can be taught as part of teacher training (Giménez, Font & Vanegas, 2013; Nilssen, 2010; Rubio, 2012; Seckel, 2016; Sun & van Es, 2015; Turner, 2012).

Teachers positively value the CDS as a tool to guide their reflection. However, what is observed is that this teaching and learning of the CDS present different levels of learning. For example, some CDS are more difficult to understand and operate correctly in class redesign, such as the epistemic CDS, a dimension valued by teachers as the most complex. The difficulty that teachers have to reflect on, understand, and operationalize this CDS is consistent with international results that teachers have difficulties interpreting epistemic aspects of tasks and identifying their educational potential (Stahnke, Schueler & Roesken-Winter, 2016).

Another result found is that teachers positively value using the professional LS teaching strategy as a context that allows them to better understand the CDS. Although the participants indicate greater difficulty in understanding the epistemic CDS, they state that they achieve greater mathematical knowledge for the teaching of the Pythagorean Theorem during the development of the course.

Furthermore, the participants highlight the time factor as a limitation for a deeper understanding of the epistemic CDS. Regarding the teaching of the CDS, its components and indicators, two aspects must be highlighted. The first is that being a consensus, they are easily assumed by teachers in their teaching processes. However, the fact that they are a consensus implies that they are ambiguous (¿for example, what does richness of processes mean? What does learning mean? What can be understood by motivation? etc.) and, to teach them, these components need to be nuanced in the development of the training course itself and, therefore, a course based on the teaching of CDS requires a greater investment of time so that these criteria are more refined and better operationalized. To this end, it is important to promote the teaching of the CDS as tools for reflection from their initial training.

According to the conclusions drawn, the study has limitations and offers opportunities for future research. One of them is that the course was applied in a specific scenario (mathematics teachers who had their training and teaching practice in Brazil). Therefore, implementing the course in a different setting and with different participants could lead to different results, which is something to consider as a possible future direction for this line of work.

Another limitation concerning this study is that the course was carried out virtually (in the context of the COVID-19 pandemic), which is one of the factors that, according to the participants themselves, prevented them from having enough time to understand CDS and discuss the class review on teaching Pythagorean Theorem. Applying for the course in face-to-face mode is an option for a future line.

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