Mathematical Understanding:
Systematic Mapping Study Based on Bibliometric Analysis

Efuansyah
Universitas Negeri Surabaya
Universitas PGRI Silampari

Agung Lukito
Universitas Negeri Surabaya

Pradnyo Wijayanti
Universitas Negeri Surabaya

This study aims to identify research trends related to mathematical understanding and research gaps using the Systematic Mapping Study (SMS) method of data collection through Publish or Perish (PoP) software to search the initial database on Scopus. Furthermore, the initial data obtained is analyzed using VOSviewer software. The number of published articles we selected according to research objectives is 23, and we reviewed them using VOSviewer. The level of education in the articles we have reviewed starts from elementary, junior high, high school, prospective teachers, and teachers. There are 4 clusters of research on mathematical understanding. In cluster 3, there is still space to conduct research related to the growth of mathematical understanding in junior high school students because, based on the results of the bibliometric analysis, there has not been much research related to the growth of mathematical understanding in junior high school students.

Keywords: bibliometrics, mathematical understanding, Scopus, publish or perish, VOSviewer

INTRODUCTION

Understanding or comprehending knowledge is necessary for teaching and learning activities. Understanding is a process of thinking and learning about knowledge. The act of understanding involves establishing meaningful connections between information within one's mind (von Glasersfeld, 1987). However, many educators tend to prioritize memorization over understanding in their teaching methods. Daud et al. (2006) discovered that conventional approaches like lectures, memorization exercises, Q&A sessions, and discussions are still widely employed by teachers. Effectively apply knowledge to new situations, comprehending the underlying meaning is crucial. Comprehension entails constructing a framework that reflects the interrelationships among various elements of a comprehended concept (Greeno, 1978, 242-243). Skemp (2006) elaborates on the dual nature of "understanding," which encompasses both relational and instrumental aspects. Relational understanding involves using information to answer the question 'why' in a meaningful way by relating it to previous information. Instrumental understanding involves achieving results quickly by learning and solving problems easily. While constructing relational information is complex and time-consuming, it is easier to remember information and transform it into different forms. In this case, students can develop different strategic solutions. Understanding is essential, especially in explicit materials such as mathematics. Kilpatrick et al. (2001: 140) state that understanding is the ability to comprehend mathematical ideas.

Mathematics is an essential and beneficial subject in life. It can be mastered well if we understand its material. Mathematics is the discipline that encompasses the logical exploration of shape, composition, quantity, and the interconnectedness of various concepts (James & James, 1976). Mathematics is considered the queen of knowledge, so it is vital to learn and understand. Even though it is a mathematical process, other general subjects require math knowledge. Mathematics is closely related to understanding, namely mathematical understanding.

Mathematical understanding can be viewed from two perspectives, encompassing both a process and a result. Both in Chinese and English, the term "understanding" has two distinct interpretations: as the present participle of the verb "to understand," signifying the process, and as a gerund functioning as a noun, indicating the result or outcome. Some researchers have adopted a process-oriented approach to assess mathematical understanding. The issue of comprehending mathematics has garnered significant attention from researchers (Masataka, 2010). Empirical evidence demonstrates that numerous students worldwide lack a solid grasp of mathematics (Cai, 2017; Li et al., 2008; Torbeyns et al., 2009). Address this challenge, mathematical understanding needs to be profound, particularly in terms of relational understanding. According to Pirie and Kieren (1992, 1994), mathematical understanding is seen as a continuous and dynamic process involving the organization of one's knowledge structure. Understanding, both philosophically and practically, is a highly intricate concept (Sierpinska, 2013). Smith (2000) highlighted that the primary objective of educational reform is to assist students in learning mathematics with comprehension. The aim of learning mathematics is for students to develop the capacity to understand the subject (NCTM, 2000).

LITERATURE REVIEW

Mathematical Understanding Concept

Exploring activities related to understanding continues to be a prominent subject in mathematics education, with a recognized requirement for improved insights into the development of mathematics learning within social environments. (Fransisco, 2012). Understanding is a crucial thing in learning mathematics because without a good understanding, learning mathematics will be complex. Hiebert & Carpenter (1992) explains many benefits of knowledge gained in learning mathematics with the following understanding: a) generative, meaning that knowledge formed from learning outcomes with the understanding that at any time can be revived (stimulated). b) meaningful, adjusting the subject matter with students' thinking abilities, enabling more meaningful learning activities. c) strengthens memory and reduces the amount of information that has to be memorized. d) facilitating the transfer of learning, the transfer occurs in learning with understanding or understanding because there are contextual similarities between the new knowledge to be learned and existing knowledge which can quickly be resurrected. e) influencing trust, students who learn with understanding will always bring up knowledge that is systematically interconnected in a cognitive structure.

Skemp (2006) classifies understanding into two types: relational and instrumental. Relational understanding refers to an individual's capacity to employ a mathematical procedure that establishes connections among different mathematical concepts. On the other hand, instrumental understanding pertains to an individual's ability to utilize a mathematical procedure to solve a problem, without necessarily comprehending the underlying reasons behind its application. In other words, students who have only instrumental understanding memorize the formula. Students need to understand mathematics sufficiently to have both relational and instrumental understanding. Furthermore, mathematical understanding is necessary for learning mathematics.

Mathematical understanding is a complex phenomenon for children who engage in it (Pirie & Kieren, 1989). It consists of two axes: (1) the vertical axis, which implies several levels of understanding such as relationships and general relationships between mathematical entities, and (2) the horizontal axis, which implies intuition, reflection, and analytical understanding that are learned at every level of understanding (Masataka, 2010).

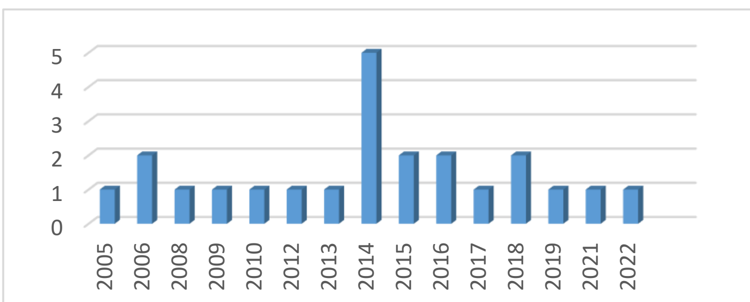
Systematic Mapping Study Design

The research questions in this study are:

1. How is the publication of mathematical understanding articles spread during the 2005-2022 period?
2. How is mathematical understanding related to other fields?
3. How is the focus of research on mathematical understanding related to educational level?
4. How is the cluster of mathematical understanding formed (based on VOSviewer analysis)?

The systematic mapping design used follows that of Agre (2015). A systematic mapping study is similar to a systematic review in that it maps topics and synthesizes study results. The process of systematic mapping offers a categorization framework for organizing and classifying published research reports and their corresponding results. The study presented here includes a related discussion of mathematical understanding. Scopus published and indexed articles from 2005 to 2022. The recent interest in researching this topic is shown by research trends studied by year of publication and presented in figure 1.

FIGURE 1
SCOPUS ARTICLE PUBLICATION MATHEMATICAL UNDERSTANDING



Inclusion, Search, and Selection

The inclusion criteria for this study were to discuss mathematical understanding in mathematics education. Mathematical understanding is an important topic to discuss with several existing studies. Research related to mathematical understanding, for example, on the growth of understanding in mathematical problems (Patmaniar, Amin & Sulaiman, 2021). In addition, understanding of mathematics with a contextual approach (Bernard, 2019), understanding of mathematics (Sierpinska, 2013), collective understanding of mathematics as improvisation (Martin et al., 2006), and the growth of students' mathematical ideas is interconnected with their behavioral patterns. (Warner, 2008).

There are many databases to search for scientific articles, but in this case, we chose the scopus database, an article that is reputable and known almost all over the world. After searching the database using publish or perish with the keywords best way to understand math (46 articles), conceptual in math (17 articles), conceptual learning in math (31 articles), conceptual math (217 articles), and conceptual understanding in math (18 articles). In addition, other keywords are also used, such as conceptual understanding in mathematics (9 articles), developing conceptual understanding in mathematics (8 articles), mathematical understanding (2 articles), NCTM conceptual understanding (2 articles), procedural understanding in math (4 articles), and understand the problem mathematics (5 articles). Furthermore, with other keywords, namely understanding basic math (5 articles), basic understanding trigonometry (1 article), understanding common core math (1 article), understanding math concepts (8 articles), understanding mathematical proof

(3 articles), understanding math (4 articles), and using & understanding mathematics (3 articles). Based on the abstract, we first screened all non-education-related publications, followed by a second round of screening which removed publications that did not apply to mathematics learning. Furthermore, 23 articles presented mathematical understanding to be analyzed and classified.

Categorization Criteria

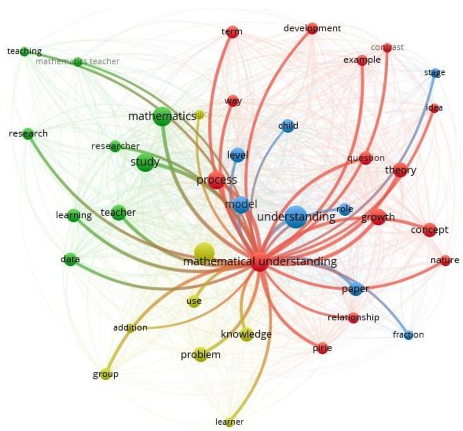
We follow the methodology presented in Webster & Watson (2014) to answer the research questions, which categorize the focus of concept-centric reviews related to mathematical understanding in learning mathematics. The article review provided us with information that made it possible to classify the current research on the following dimensions:

- Mathematical understanding
- Context
- Level of education
- Application

Mathematical Understanding

Concerning the categorization of mathematical understanding, we first reviewed a literature review of published, conceptual, and existing literature on mathematical understanding (excluding 23 articles reporting empirical research). However, we found that there has yet to be a general agreement regarding mathematical understanding. Pirie & Kieren (1989) stated that mathematical understanding is a complex phenomenon for children who do it. Mathematical understanding is a dynamic mind activity (Koyama, 1993). Something slightly different was expressed by Martin et al. (2006), collective mathematical understanding is a process of creative improvisation that emerges, illustrating how this can be observed in action. Mathematical understanding is influenced by the presence of manipulatives within the learning environment, along with the utilization of tasks incorporating verbal, graphic, and algebraic representations, contributes to the educational experience. (Gülkilika et al., 2015). Mathematical understanding has a different meaning because it is related to relationships in other fields, as shown in Figure 2.

FIGURE 2
RELATIONSHIP MATHEMATICAL UNDERSTANDING AND OTHER FIELDS



Context

The context used in various studies in the articles we have obtained is related to utilization of visual representations is instrumental in improving the understanding of mathematical notation among college students. It states that the context is represented visually in fractions to assist in interpreting what is meant to represent fractions (Atagi, 2016), context is a mathematical activity (Pirie, 2000). An understanding of a particular context through a seemingly more remote real-world context (Cai & Ding, 2017). The context of the everyday classroom life of prospective teachers seems necessary if prospective teachers can construct an understanding to develop new techniques (Borgen, 2006). In addition, using percentage as an operator in the context of money by using a double number line as a representation (Wright, 2014) and mathematics is presented in the context of problems where real solutions are needed (Martin et al., 2005).

Level of Education

At the educational level, the articles we reviewed came from different subjects, such as elementary school, which was researched by Mcneil (2014), stating that most elementary school students in the United States find it difficult to understand mathematical equality in symbolic terms. The study was conducted among 61 first- and second-graders in public and private elementary schools in urban Midwestern U.S. cities (Fyfe, 2015). In addition, the sixth-grade students in a public primary school in Turkey conducted a study (Gokkalp & Bulut, 2018). Data were collected in the third grade of national primary schools (Masataka, 2010). Additionally, sixth graders in Canada faced issues in the parallel ethics domain of computing (Martin et al., 2006). The authors report that three students who were diagnosed with EBD while in college improved three tasks when they were more likely to work. (Tan, 2016).

Furthermore, a case study of 3 weeks of instruction with eight students aged 12 and 13 has provided data (Wright, 2014). Elementary school students' mathematics learning with CIM, CIH, and CPN shows that it contributes to students' growing mathematical understanding (Martin & Towers, 2015). The psychological organization of the second type of children reflects more conceptual groups, and it has a better understanding of mathematical equivalent (Chesney et al., 2014).

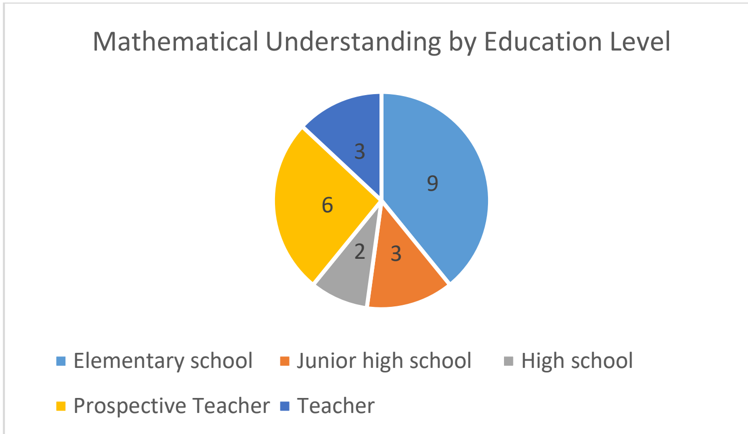
In the research of Ibili et al., In junior high school (2019) shows that about 70 AR apprentices have been developed to convey the key element of the eighth-grade geometric curriculum, one of which is Tomas (14 years). He gave signs of further development of technology. In the independent resettlement of independent student Miguel Angel (14 years old) and Matias (12), he was better and exquisite Essence On the other hand, Analysis is based on case studies. Middle School students participate in different abilities. This case examines combinations of postoperative problems and problem solutions (Warner, 2008). Furthermore, Sengual et al. (2017) secondary school students' mathematical understanding of different variables.

In high school, as in the study, four grade 10 observe student translations, rotations, reflections, and expansions in class (Gülkılıka et al., 2015). Other research examines high school students' knowledge when they use backtracking to solve math problems (Patmaniar, Amin & Sulaiman, 2021).

For prospective teachers, research by Towers and Martin (2014) explored the mechanism of a group of prospective teachers who worked on math problems together. We explored entry-level trainee John's understanding of multiplication, fractions, and imperial units of measure as he tried to solve problems (Martin et al., 2005). In addition, we provided a detailed account of how students' comprehension of numerical series evolves through their participation in math class assignments (Codes et al., 2013). On the other hand, Holm and Kajander (2012) explored pre-service teachers' initial understanding of mathematics and how this understanding is developed during mathematics training for upper-grade elementary school teachers. Additionally, we collaborated with students who were in the process of becoming primary school teachers (Martin & Towers, 2009). The performance of students showed a notable improvement when they were presented with accurate visuals compared to misleading visuals or no visuals at all (Atagi et al., 2016).

In teachers, the assessment of Oudman et al. (2018) stated that by controlling the availability of information that can be used for assessment, the accuracy with which teachers can assess students' conceptual mathematical understanding can be improved. In another study, Chinese mathematics teachers believed that comprehension is a network of connections, the result of continuous connections (Cai & Ding, 2017). For the current study, our specific choice of two teachers was driven by our observations, which revealed that both educators exhibited a conceptual orientation. Notably, they actively fostered an environment where students were encouraged to openly share their mathematical thinking in class and warmly welcomed any questions they had (Reinke et al., 2022). The visualization of mathematical understanding based on the educational level in the articles we studied is shown in figure 3.

FIGURE 3
VISUALIZATION OF MATHEMATICAL UNDERSTANDING BASED ON EDUCATIONAL LEVEL



Application

Table 1 presents data related to authors, article titles, journal names, journal publishing countries, and research focus. We see that the 23 articles we have examined focus on the study of mathematical understanding from the elementary, junior high, high school, prospective teachers, and teacher levels. Various methods were carried out in research, such as folding back (Pirie Kieren model), using applications, teaching lenses, theoretical perspectives, NDP teaching, and potential and practical algorithm procedures. Related to the material discussed in the research of the 23 articles we have reviewed are material on quadratic equations, percentages, geometry, fractions, geometric transformations, number properties, arithmetic series, number patterns, and multiplication of natural numbers. More details can be seen in Table 1.

TABLE 1
ARTICLES AND RESEARCH FOCUS

Author	Article Title	Journal	Country	Research Focus on Mathematical Understanding
Reinke, Stephan, Civak, & Casto (2022)	Teachers' press for contextualization to ground students' mathematical understanding of ratio	Journal of Mathematics Teacher Education	Netherlands	We recognize the importance of a particular teacher movement that is promoting a contextualized approach aimed at explicitly grounding students' use of representations on their commonsense understandings problem context.
Patmaniar, Amin, & Sulaiman, (2021)	Students' growing understanding in solving mathema-tics problems based on gender: elaborating folding back	Journal on Mathematics Education	Indonesia	Assessing the level of knowledge of high school students in solving math problems using the folding back method.
Ibili, Resnyansky, & Billinghurst (2019)	Applying the technology acceptance model to understand math teachers' perceptions towards	Education and information Technologies	Turkey & Australia	The study aimed to evaluate the acceptance rate of math teachers and investigate the effectiveness of the Augmented Reality Geometry Tutorial System (ARGTS), a mobile application utilizing augmented reality

Author	Article Title	Journal	Country	Research Focus on Mathematical Understanding
	an augmented reality tutoring system			(AR) technology. The purpose was to enhance students' 3D geometric reasoning skills.
Gokalp & Bulut (2018)	A New Form of Understanding Maps: Multiple Representations with Pirie and Kieren Model of Understanding	International Journal of Innovation in Science and Mathematics Education	Turkey	The objective of the study was to enhance the mapping capabilities of the Pirie and Kieren models by exploring the incorporation of multiple representations, with the aim of improving the conceptual understanding and performance in rendering mathematical concepts.
Oudman, Van de Pol, Bakker, Moerbeek, & Van Gog (2018)	Effects of different cue types on the accuracy of primary school teachers' judgments of students' mathematical understanding	Teaching and Teacher Education	The Netherlands	By examining the availability effect cuetype, the study sought to gain a deeper understanding of how teacher assessment accuracy can be enhanced.
Cai & Ding (2017)	On mathematical understanding: perspectives of experienced Chinese mathematics teachers	Journal of Mathematics Teacher Education	United States	Investigate the views of experienced Chinese mathematics teachers on understanding mathematics.
Tan (2016)	Developing mathematical understanding and students with emotional and behavioural disorders: a review of the literature	Emotional and Behavioural Difficulties	United States	The study examined existing mathematics research interventions for students with EBD using a pedagogical understanding lens.
Atagi, Dewolf, Stigler, & Johnson (2016)	The role of visual representations in college students' understanding of mathematical notation	Journal of Experimental Psychology: Applied	United States	Develop an understanding of fractions involving the relationship between non-symbolic visual representations and symbolic representations.
Fyfe, McNeil, & Borjas (2015)	Benefits of "concreteness fading" for children's mathematics understanding	Learning and Instruction	United States	Exploring concrete fading techniques which involve transitioning from concrete materials to abstract symbols, educators can facilitate children in expanding their knowledge beyond conventional teaching methods.
Gulkilik, Ugurlu, & Yuruk (2015)	Examining Students' Mathematical Understanding of Geometric Transformations	Kuram Ve Uygulamada Egitim Bilimleri	Turkey	The study analyzed grade X Students' mathematical comprehension of geometric transformations progressed within a reliable and supportive environment that incorporated multiple representations.

Author	Article Title	Journal	Country	Research Focus on Mathematical Understanding
	Using the Pirie-Kieren Model.			
McNeil (2014)	A change-resistance account of children's difficulties understanding mathematical equivalence	Child Development Perspectives	United States	The role of the early learning environment in shaping and hindering developmental misunderstandings enhances understanding of fundamental mental processes involved in mathematical reasoning
Towers & Martin (2014)	Building Mathematical Understanding Through Collective Property Noticing	Canadian Journal of Science, Mathematics and Technology Education	Canada	A group of trained teachers work together to teach math concepts by combining individual contributions to create a collective image.
Lyndon, Martin & Towers (2014)	Growing mathematical understanding through Collective Image Making, Collective Image Having, and Collective Property Noticing	Educational Studies in Mathematics	Canada	Formulate theoretical frameworks aimed at describing and elucidating the development of collective mathematical understanding.
Wright (2014)	Frequencies as proportions: Using a teaching model based on Pirie and Kieren's model of mathematical understanding	Mathematics Education Research Journal	Australia	The NDP teaching model can be flexibly applied to teaching with appropriate attention to desired outcomes of action through images and the use of images to develop general numerical attributes.
Chesney, McNeil, Matthews, Byrd, Petersen, Wheeler, Fyfe, & Dunwiddie (2014)	Organization matters: Mental organization of addition knowledge relates to understanding math equivalence in symbolic form	Cognitive Development	United States	Children tend to mentally organize their additional knowledge around conceptually related groupings. This helps them develop a better understanding of mathematical equivalence.
Codes, Astudillo, Martin, & Perez (2013)	Growth in the understanding of infinite numerical series: a glance through the Pirie and Kieren theory	International Journal of Mathematical Education in Science and Technology	Spain	During a mathematical educational task related to the concept of sequence of numbers, students' mathematical understanding develops as they learn how to identify patterns and relationships between numbers.
Holm & Kajander (2012)	'I Finally Get It!': Developing mathematical understanding during teacher education	International Journal of Mathematical Education in Science and Technology	Canada	Exploring pre-service teachers' initial understanding of mathematics and how understanding is developed during mathematics training for upper-grade elementary school teachers.

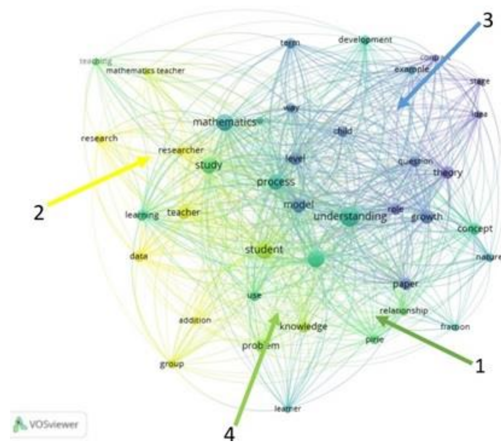
Author	Article Title	Journal	Country	Research Focus on Mathematical Understanding
Koyama (2010)	A Research On The Validity And Effectiveness Of the "Two-Axes Process Model" Of Understanding Mathematics At the Elementary School Level	Hiroshima journal of mathematics education	Japan	Test the validity and effectiveness of biaxial understandings of mathematical process models as useful and effective frameworks for mathematics teachers means to evaluate whether these models are helpful for teachers in teaching mathematics
Martin & Towers (2009)	Improvisational coactions and the growth of collective mathematical understanding	Research in Mathematics Education	United States	Collective mathematical understanding is when people work together to understand math better.
Warner (2008)	How do students' behaviors relate to the growth of their mathematical ideas?	The Journal of Mathemati-cal Behavior	United States	Examine the correlation between student behavior and the developmental progression of mathematical ideas. thought (using the Pirie-Kieren model).
Romero & Mari (2006)	Assessing understanding in mathematics: steps towards an operative model	For the learning of Mathematics	Spain	The complexity of the diagnosis and assessment of understanding in mathematics and the potential and practical application of written algorithmic procedures for multiplying natural numbers.
Martin, Towers, & Pirie (2006)	Collective Mathematical Understanding as Improvisation	Mathemati-cal Thinking and Learning	United States	Exploring how people understand math and how they can work together to understand it better
Martin, LaCroix, & Fownes (2005)	Folding Back and the Growth of Mathematical Understanding in Workplace Training.	Adults learning mathematics an international journal	United States, Canada, & Canada	An exploration of the growth of understanding of mathematics in various construction trade training programs.

Mapping Study Results

This section outlines the distribution of published works based on each classification criterion. It is based on the results of the Vosviewer analysis in Figure 3.

In Figure 4, the VOSviewer visualization shows the clustering of mathematical understanding research in learning mathematics. Cluster 1 (dark green), related to components of mathematical understanding that are integrated or identified as mathematics learning products, such as mathematical equality, can be overcome by understanding concepts (McNeil, 2014). A contrast is when students in abstract conditions are asked to write down math problems (Fyfe, McNeil, & Borjas, 2015). Understanding grows through a dynamic and active process that encompasses the development of action (Patmaniar, Amin, & Sulaiman, 2021). When students understand geometric shapes, it helps them understand math better (Gulkilik, Ugurlu, & Yuruk, 2015). Understanding the math helps students solve problems correctly and vice versa (Tan, 2016). When students don't know something, it can make it harder to learn related concepts (Wright, 2014). If students think about their own ideas while helping a friend with math problems, they can improve their own understanding (Warner, 2008).

FIGURE 4
RESEARCH CLUSTERING



Cluster 2 (yellow) is related to the prospective teacher's mathematical understanding knowledge related to assessment and the types of actions that are effective in learning, namely the types of actions and learning mathematics that are visible and occur when groups of students work together on some mathematics material (Martin & Towers, 2009). The accuracy of teacher assessments in the meta-analysis of teacher assessments was measured by asking teachers about student ratings (Oudman et al., 2018). Subject matter knowledge for mathematics teachers has been the concern of many researchers in trying to produce effective elementary school teacher candidates (Holm & Kajander, 2012).

In cluster 3 (blue), students' mathematical understanding of a topic can be identified by understanding each stage of their understanding. Many students need help understanding mathematical similarities (McNeil, 2014). The relationship between students' preferences in using various representations and the achievement level of understanding the multiplication of fractions (Gokalp & Bulut, 2018). On the other hand, there are three different levels to consider mathematics in the work world: skill level, understanding level, and identity level (Martin, LaCroix, & Fownes, 2005). The Pirie-Kieren theory offers a framework for examining the involvement of participants in revisiting prior understandings during the process of comprehending mathematics (Patmaniar, Amin, & Sulaiman, 2021). Within this cluster, there remains an opportunity to conduct research focusing on the development of mathematical understanding among junior high school students. Based on the analysis, there is a limited amount of research pertaining to the growth of mathematical understanding specifically in this student demographic.

In cluster 4 (light green) mathematical understanding is related to understanding concepts, namely children who can mentally gain additional conceptual knowledge related to grouping numbers will more easily understand mathematical equations (Chesney, McNeil, Matthews, Byrd, Petersen, Wheeler, Fyfe, & Dunwiddie, 2014). Cai and Ding (2017) stated that mathematical understanding is an ongoing and dynamic process involving the organization of knowledge structures. Wright (2014:105) suggests that the numeracy development program (NDP) instructional model operates under the premise that teacher actions play a crucial role in guiding learners' advancement through various levels of mathematical understanding. The model consists of three layers: use of materials, use of images, and use of digital attributes, emphasizing appropriate representation choices for teachers and students using images and attributes. It uses levels of numerical attributes and asks students questions in symbolic language (Wright, 2014). In addition, questions have been prepared to check students' initial comprehension level can be assessed by evaluating their capacity to apply concepts related to oral, graphical, and mathematical algebraic representations in the context of geometric transformations (Gulkilik, Ugurlu, & Yuruk, 2015).

CONCLUSION

Based on the results of an analysis of the distribution of Scopus article publications related to mathematical understanding during the 2005-2022 period, mathematical understanding is still a research trend. The number of publications we have selected according to research objectives is 23. We reviewed them using VOSviewer, the word mathematical understanding is related to conceptual knowledge, mathematical problems, mathematics students, material for addition, multiplication, division, number patterns, percentages, fractions, geometric transformations, quadratic equations, triangles, arithmetic, and mathematical notation. The level of education raised in the research that we have examined starts from elementary school with nine articles, junior high school with three articles, high school with two articles, prospective teachers with six articles, and teachers with two articles. There are 4 clusters of research on mathematical understanding. Namely, cluster 1 is related to components of mathematical understanding that are integrated or identified as products of learning mathematics. Cluster 2 relates to prospective teachers' mathematical understanding knowledge related to assessment and types of actions that are effective in learning; cluster 3 relates to students' mathematical understanding of a material at each layer of understanding; Cluster 4 mathematical understanding relates to conceptual understanding. In cluster 3, based on the findings of bibliometric analysis, limited research exists on the development of mathematical understanding among junior high school students, indicating a gap in current knowledge. Consequently, there is an opportunity for further research in this area to enhance our understanding of how mathematical understanding evolves in students at the junior high school level.

ACKNOWLEDGMENTS

The authors express their gratitude to the Center for Education Financial Services (Pusat Layanan Pembiayaan Pendidikan) and the Education Fund Management Institution (Lembaga Pengelola Dana Pendidikan) for their support, as well as to the School of Postgraduate Studies.

REFERENCES

Agre, G. (2015). Gamification in Education: A Systematic Mapping Study. *Educational Technology & Society*, pp. 75–88. Retrieved from https://www.jstor.org/stable/jeductechsoci.18.3.75

Atagi, N. (2016). The role of visual representations in college students' understanding of mathematical notation. *Journal of Experimental Psychology: Applied*, *22*(3), 295–304. https://doi.org/10.1037/xap0000090

Bernard, M. (2019). Improve the ability of understanding mathematics and confidence of elementary school students with a contextual approach using VBA learning media for Microsoft Excel. In *Journal of Physics: Conference Series* (Vol. 1318, Issue 1). https://doi.org/10.1088/1742-6596/1318/1/012035

Borgen, K.L. (2006). *From mathematics learner to mathematics teacher: Preservice teachers' growth of understanding of teaching and learning mathematics* [Doctoral dissertation, University of British Columbia]. https://doi.org/10.14288/1.0055178

Cai, J., & Ding, M. (2017). On Mathematical Understanding: Perspectives of experienced liChinese mathematics teachers. *Journal of Mathematics Teacher Education*, *20*(1), 5–29. https://doi.org/10.1007/s10857-015-9325-8

Chesney, D.L., McNeil, N.M., Matthews, P.G., Byrd, C.E., Petersen, L.A., Wheeler, M.C., . . . Dunwiddie, A.E. (2014). Organization matters: Mental organization of addition knowledge relates to understanding math equivalence in symbolic form. *Cognitive Development*, *30*(1), 30–46. https://doi.org/10.1016/j.cogdev.2014.01.001

Codes, M., Astudillo, M.T.G., Martín, M.L.D., & Pérez, M.C.M. (2013). Growth in the understanding of infinite numerical series: A glance through the Pirie and Kieren theory. *International Journal of Mathematical Education in Science and Technology*, *44*, 652–662. https://doi.org/10.1080/0020739X.2013.781690

Daud, M.Y., Ahmad, M., & Lubis, M.A. (2006). *Penggunaan Internet dalam Aktiviti Pengajaran dan Pembelajaran Pendidikan Islam.*

Daud, M.Y., Ahmad, M., & Lubis, M.A. (2006). Penggunaan Internet dalam Aktiviti Pengajaran dan Pembelajaran Pendidikan Islam. *Dlm. Ab. Halim Tamuri, Nik Mohd Rahimi Nik Yusoff, Aliza Alias, Saemah Rahman, Zamri Mahamod, Mazalah Ahmad, Siti Fatimah Mohd. Yassin, Maimun Aqsha Lubis, Faridah Mydin Kytty, A.S Sulaima (pnyt). Wacana Pendidikan Islam (Siri 5) Pendidikan Islam & Bahasa Arab Pemangkin Peradaban Ummah, hl, 129-136.*

Francisco, J.M. (2012). Learning in collaborative settings: Students building on each other's ideas to promote their mathematical understanding. *Educational Studies in Mathematics*, *82*, 417–438. http://doi.org/10.1007/s10649-012-9437-3

Fyfe, E. (2015). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction*, *35*, 104–120. https://doi.org/10.1016/ j.learninstruc.2014.10.004

Gokalp, N.D., & Bulut, S. (2018). A New Form of Understanding Maps: Multiple Representations with Pirie and Kieren Model of Understanding. *International Journal of Innovation in Science and Mathematics Education*, *26*. Retrieved from https://www.semanticscholar.org/paper/5063da975412e3c04cc100983278ee431c02f57c

Greeno, J.G. 1978. *Natures of Problem-Solving Abilities*. In W.K. Estes (Ed.), *Handbook of Learning and Cognitive Processes* (Volume 5). Human Information Processing. New Jersey: Lawrence Erlbaum Associates.

Gülkilika, H., Uğurlu, H. H., & Yürük, N. (2015). Examining Students' Mathematical Understanding of Geometric Transformations Using the Pirie-Kieren Model. *Kuram Ve Uygulamada Egitim Bilimleri*, *15*, 1531–1548. https://doi.org/10.12738/estp.2015.6.0056

Hiebert, J., & Carpenter, T.P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65–97). Macmillan Publishing Co, Inc.

Holm, J. (2012). "I Finally Get It!": Developing mathematical understanding during teacher education. *International Journal of Mathematical Education in Science and Technology*, *43*(5), 563–574. https://doi.org/10.1080/0020739X.2011.622804

James, G., & James, R.C. (1976). *Mathematic Dictionary*. New York: van Nostrand Rienhold.

Kilpatrick, J., Swafford, J. & Findell, B. (Eds). (2001). *Adding it up: Helping children learn mathematics* (pp. 115–155). Washington: National Academy Press.

Koyama, M. (1993). Building a Two Axes Process Model of Understanding Mathematics. *Journal on Mathematics Education*, *1*, 63–73. Retrieved from https://www.semanticscholar.org/paper/ 96c6b9113c2a3f85874a4a4f33cf1ab6acf656cb

Martin, L., Lacroix, L.N., & Fownes, L. (2005). Folding Back and the Growth of Mathematical Understanding in Workplace Training. *ALM International Journal*, *1*(1), 19–35. Retrieved from https://www.semanticscholar.org/paper/1192709d3c69dfd8cc54c6298a 2067012581d82c

Martin, L., Towers, J., & Pirie, S. (2006). Collective Mathematical Understanding as Improvisation. *Mathematical Thinking and Learning*, *8*, 149–183. https://doi.org/10.1207/s15327833mtl0802_3

Masataka. (2010). A Research On The Validity And Effectiveness Of "Two-Axes Process Model" Of Understanding Mathematics At Elementary School Level. *Hiroshima Journal of Mathematics Education*, *9*, 1–9. Retrieved from https://www.semanticscholar.org/paper/f6a7bf34064080cf57f821704181dabf479e577

Mcneil, N. (2014). A Change-resistance Account of Children's Difficulties Understanding Mathematical Equivalence. *Child Development Perspectives*, *8*(1), 42–47. https://doi.org/10.1111/cdep.12062

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Oudman, S. (2018). Effects of Different Cue Types on the Accuracy of Primary School Teachers' Judgments of Students' Mathematical Understanding. *Teaching and Teacher Education*, *76*, 214–226. https://doi.org/10.1016/j.tate.2018.02.007

Patmaniar, P., Amin, S.M., & Sulaiman, R. (2021). Students' Growing Understanding In Solving Mathematics Problems Based On Gender: Elaborating Folding Back. *Journal on Mathematics Education*, *12*(3), 507–530. https://doi.org/10.22342/jme.12.3.14267.507-530

Pirie, S. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal*, *12*(2), 127–146. https://doi.org/10.1007/ BF03217080

Pirie, S., & Kieren, T. (1989). A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics*, *9*, 7–11. Retrieved from https://www.semanticscholar.org/paper/1759cf5ff70c072bbf49aaf0cd60d9fdd5a05c34

Pirie, S., & Kieren, T. (1992). Creating Constructivist Environments and Constructing Creative Mathematics. *Educational Studies in Mathematics*, *23*, 505–528. Retrieved from https://www.jstor.org/stable/3482850.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165–190. Retrieved from https://www.jstor.org/stable/3482783.

Reinke, L.T., Stephan, M., Ayan, R., & Amanda, C. (2022). Teachers' press for contextualization to ground students' mathematical understanding of ratio. *Journal of Mathematics Teacher Education*, *0123456789*. https://doi.org/10.1007/s10857-022-09531-w

Şengül, S., Kaba, Y., & Erdoğan, F. (2017). The Analysis of Middle School Students' Mathematical Understanding in Terms of Different Variables. *Kastamonu Egitim Dergisi*, (4), 1421–1434. Retrieved from https://www.semanticscholar.org/paper/a873ee36b84d40 cc500d1677cb0a035b84b22d92

Sierpinska, A. (2013). *Understanding in Mathematics*. London: School of Education University of Exeter. https://doi.org/10.4324/9780203454183.

Skemp, R.R. (2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School*, *12*(2), 88–95. http://doi.org/10.5951/MTMS.12.2.0088

Smith, M.S. (2000). Redefining success in mathematics teaching and learning. *Mathematics Teaching in the Middle School*, *5*(6), 378–386. http://doi.org/10.5951/MTMS.5.6.0378

Tan, P. (2016). Developing mathematical understanding and students with emotional and behavioural disorders: A review of the literature. *Emotional and Behavioural Difficulties*, *21*(4), 361–376. https://doi.org/10.1080/ 13632752.2016.1201639

Towers, J., & Martin, L. (2014). Building Mathematical Understanding Through Collective Property Noticing. *Canadian Journal of Science, Mathematics and Technology Education*, *14*, 58–75. https://doi.org/10.1080/14926156.2014.874612

von Glasersfeld, E. (1987). Learning as a constructive activity. *Proceedings of the 5th Annual Meeting of the North American Group of Psychology in Mathematics Education.* Retrieved from http://vonglasersfeld.com/083

Warner, L.B. (2008). How do students' behaviors relate to the growth of their mathematical ideas? *The Journal of Mathematical Behavior*, *27*, 206–227. https://doi.org/10.1016/J.JMATHB.2008.07.002

Webster, J., & Watson, R.T. (2014). Analyzing The Past to Prepare for The Future: Writing A Review. *MIS Quarterly*, *26*(2). http://doi.org/10.2307/4132319

Wright, V. (2014). Frequencies as proportions: Using a teaching model based on Pirie and Kieren's model of mathematical understanding. *Mathematics Education Research Journal*, *26*, 101–128. https://doi.org/10.1007/S13394-014-0118-7